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Relativistic Action at a Distance: Classical and Quantum Aspects

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Editor
Josep Llosa
Departament de Física Teòrica
Facultat de Física, Universidad de Barcelona
Diagonal, 645, Barcelona – 28, Spain

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PREFACE

This volume contains the main papers from the workshop on "Relativistic Action at a Distance: Classical and Quantum Aspects", held in Barcelona, June 15-21, 1981, in which we have tried to review the work and progress in this field over the last twenty years. One of the main aims of the meeting was to encourage communication and discussion among physicists who are working in this subject from different approaches. Our intention was to break the isolation that scientists from various lines of work often find themselves in, and who thus have often ignored each other.

Relativistic action-at-a-distance dynamics is at the present time a rather heterodox approach to interacting particle systems, particularly in the present panorama of theoretical physics clearly dominated by field theories. The reasons for this prevalence are obvious. We have, first, the success of Maxwell's electromagnetic field theory where the various attempts of Newtonian action-at-a-distance theories failed. Secondly, Maxwell's field theory supplies a suitable framework to account for radiation phenomena which, apparently, cannot be explained in the framework of an action-at-a-distance theory - actually, the first theory which succeeded in accounting for electromagnetic radiation was Wheeler and Feynman's electrodynamics (1949).

Another important factor has been the apparent incompatibility between Poincaré invariance and instantaneous action at a distance which has been often identified with "interaction transmitted at infinite speed between simultaneous states of the particles".

Nevertheless, in spite of their many successes, some important problems of field theories remain unsolved. Think, for instance, of the interaction of a particle with its own field - self-interaction. The energy associated with it is infinite, a situation which is obviously absent in an action-at-a-distance theory. This difficulty is insurmountable within classical field theories, and the renormalization techniques of QFT give satisfactory solutions in certain cases (electro-weak, QCD), but quantum gravitation still cannot be renormalized. Also there is the remarkable fact that, in dealing with bound states, quantum field theory leads to Bette-Salpeter equations quite similar to the equations that one hopes to deal with in an action-at-a-distance quantum theory.

During the last three decades a revival of action-at-a-distance theories has taken place which has taken relativistic invariance into account. A good review of the first half of this period is supplied by the reprint collection edited by E. Kerner, The Theory of Action at a

Distance in Relativistic Particle Dynamics (1972). At that time several branches had already appeared which dealt with the subject: on one hand, the non-instantaneous action-at-a-distance theories - Wheeler and Feynman's electrodynamics (1949) and the more general Van Dam and Wigner's theory (1965) - and, on the other hand, the instantaneous action-at-a-distance theories. This last group can be further divided into two subbranches: Dirac's Hamiltonian formalism (1949) and its subsequent development, and predictive relativistic mechanics. Whereas the first formalism plans the construction of a canonical realization of the Poincaré group on the phase space of the system of particles, the second emphasizes the fact that the world line of each particle must be Poincaré invariant. The no-interaction theorems must be placed between both subbranches. These theorems state that conditions a) canonical realization of the Poincaré group and b) considering the positions of the particles as canonical coordinates are uniquely compatible in the special case of free-particle systems.

Since the collection compiled by Kerner was published, these branches have undergone further development. Moreover, new approaches have also arisen, inspired either by Dirac's constraint Hamiltonian formalism (Lectures on Quantum Mechanics, 1964) or by the singular Lagrangian one. Both formalisms are canonical and manifestly covariant. The negative implications of the no-interaction theorems are avoided by dealing not with the whole phase space TM_4^N but with a sub-manifold of it.

After three decades of this relativistic revival, the action-at-a-distance approach has not achieved the degree of development of field theories, and a lot of work is still to be done. A first quantization of action-at-a-distance theories presents some difficulties which have not yet been completely understood, while second quantization must be continued from the embryonic stage presented by Professor Droz-Vincent in these proceedings, etc. The following conclusion, however, can already be drawn: the situation is not one of "action-at-a-distance" versus "field" theories; rather, both approaches must be considered as complementary tools for achieving a deeper understanding of interaction phenomena.

In the Barcelona workshop a review of these developments was presented and, as can be seen by a glance at the contents of this volume, we center our attention on "instantaneous action-at-a-distance" theories. This volume includes, besides the papers presented at the meeting, two contributions by Professor R. Arens, who had been invited to lecture but was prevented from attending the workshop. In his first paper Professor Arens presents a way of generating solutions to the Droz-Vincent's equations by means of diffeomorphisms of the phase space of a free-

particle system. In the second, he proves the existence of interacting two-particle systems under the assumptions of symmetry, dilation and Poincaré invariance.

The contribution of Professor R.N. Hill, who was one of the pioneers of predictive relativistic mechanics, deals with the origins of his approach. Part of the further work in this branch is presented in two lectures by Professor P. Droz-Vincent. The first treats the multi-time covariant formalism and the a priori Hamiltonian approach, where the result of the no-interaction theorems are avoided by giving up the canonical character of the position coordinates. The second lecture presents an attempt to second quantize a system of interacting particles. Closing the set of conferences which we have classified as predictive relativistic mechanics, Professor L. Bel's lecture deals with retarded equations and states that equations of this kind predictivize spontaneously in the cases of constant retardation and of electromagnetic and gravitational interactions.

From another point of view Professor F. Coester presents a Hamiltonian formalism which could be classified in the line of Foldy's previous work. In the first lecture the general framework is presented and, in the second, it is applied to some particular scattering problems.

The singular Lagrangian theory applied to relativistic particle dynamics is presented in this volume by Professor G. Longhi, who details the study of a two-particle system in this framework. He also discusses the problem of the separability of the interactions for two- and three-particle systems.

Finally, the contributions of Professor F. Rohrlich and of Professor I. Todorov envisage the problem of relativistic interacting N-particle systems from the viewpoint of the constrained Hamiltonian formalism. The method developed by the first author is based upon the fact that dealing with a covariant model implies the introduction of 2N spurious degrees of freedom which must be eliminated afterward. This elimination is carried out by means of 2N second-class constraints; N amongst them define the general mass shell and the remaining N constraints depend on a scalar parameter and are called fixations because they determine a particular motion of the system.

Professor Todorov's method uses the same constraint formalism, emphasizing the geometric invariants though. So, in this approach an important role in the definition of the physical phase space is assigned to the geometrical trajectory of the system, i.e., considered independently of any parametrization. This physical phase space is defined by the foliation that the world surfaces introduced onto the general mass shell.

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Barcelona, March 1982 J. Llosa

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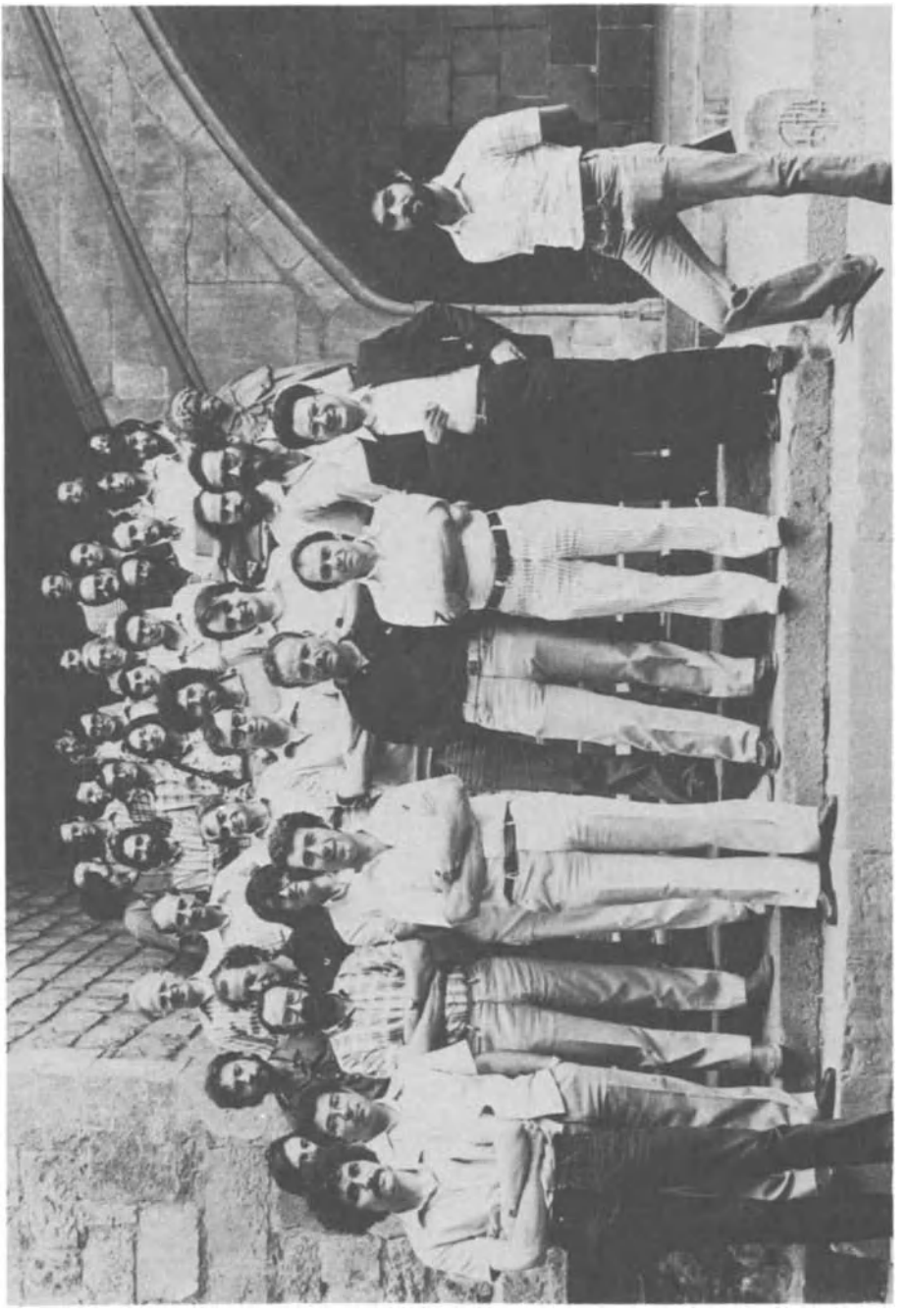

LIST OF PARTICIPANTS

AGUIRREGABIRIA, J.M., Universidad Pais Vasco
BEL, L., Institut Henri Poincaré, France
BETTE, A., University of Stockholm, Sweden
BISWAS, T., Syracuse University, U.S.A.
BONA, C., Universitat Autònoma, Barcelona
CAWLEY, R., Naval Surf. Weapons C., U.S.A.
CERVERO, ., Universidad Salamanca
CHAMORRO, A., Universidad Pais Vasco
CHELKOWSKI, S., Université Tiri Duron, Algerie
CHOW, T., California State College, U.S.A.
CIUTAT, J., Universitat Valencia
COESTER, F., Argonne National Lab., U.S.A.
COMELLES, F., Universitat Autònoma, Barcelona
CRATER, H., University of Tennessee, U.S.A.
DOMINICI, D., Ist. Naz. Fisica Nucleare, Firenze, Italy
DROZ-VINCENT, P., Collège de France, France
EDER, E., Max-Planck Inst. Astro., Austria
ELIZALDE, E., Universitat de Barcelona
FAYOS, F., Universitat de Barcelona
FUSTERO, X., Universitat Autònoma, Barcelona
GOMIS, J., Universitat de Barcelona
GRACIA BONDIA, J., Universidad de Costa Rica, Costa Rica
GRAELLS, J., Universitat de Barcelona
HEMION, G., University Bielefeld, Germany
HILL, R.N., University Delaware, U.S.A.
IBORT, L.A., Universidad de Zaragoza
IRANZO, V., Universitat de Barcelona
KAMIMURA, K., Ist. Naz. Fisica Nucl.,Firenze, Italy
LAPIEDRA, R., Universidad de Santander
LLOSA, J., Universitat de Barcelona
LOBO, J.A., Universitat de Barcelona
LONGHI, G., Ist. Naz. Fisica Nucleare, Firenze, Italy
LUSANNA, L., Ist. Naz. Fisica Nucleare, Firenze, Italy
MARQUES, F., Universitat de Barcelona
MARTIN, C., Universitat de Barcelona
MARTIN, J., Universidad de Salamanca
MAS, L., Universitat Palma de Mallorca
MOLINA, A., Universitat de Barcelona
NAKA, S., Nihon University, Japan

NIETENDEL, J., Educational University , Lesna, Poland
PARRA, J.M., Universitat de Barcelona
PAURI, M., Universita di Parma, Italy
POCH, A., Universitat de Barcelona
PONS, J., Universitat de Barcelona
PORTILLA, M., Universitat Valencia
RAFANELLI, K., University New York, U.S.A.
RAÑADA, M.F., Universidad de Zaragoza
REDONDO, J.M., Virginia Polytec. Inst. U.S.A.
ROHRLICH, F., Syracuse University, U.S.A.
RUIZ, E., Universidad de Salamanca
SALAS, A., Universitat Autònoma, Barcelona
SANCHEZ, N., Universitat Barcelona
SANCHEZ-RON, J.M., Universidad Autónoma, Madrid
SADJIAN, H., Université Paris XI, France
SORACE, E., Ist. Naz. Fisica Nucleare, Firenze, Italy
SUCHANED, R., Educational University, Lesna, Poland
TODOROV, I., Inst. Nuclear Research, Sofia, Bulgaria.

2-PARTICLE INTERACTIONS PRODUCED BY TRANSFORMATIONS OF PHASE SPACE

Richard Arens

Mathematics Department
University of California, Los Angeles
Los Angeles, CA 90024

1. Introduction

The motions of a pair of interacting particles are given by a set M of 2-dimensional submanifolds of 16-dimensional cartesian space $\mathbb{R}^{16}$ (which we took the liberty of calling phase space in the title). If the particles are in fact non-interacting, then these motions form a set M_o of flat 2-dimensional submanifolds.

The idea is to transform $\mathbb{R}^{16}$ (more precisely an appropriate subset of it) onto itself so as to transform M_o into a class M of 2-dimensional submanifolds capable of being regarded as the motions of a non-zero, Poincaré-invariant interaction.

Necessary and sufficient properties for a map φ to produce such an interaction are presented. It is proved that all interactions so obtained have this property: if the particles appear to be at rest for some observer, then they remain at rest for that observer (and thus have parallel straight world lines for that motion).

2. N particle interactions

Let $T^1(\mathbb{R}^4)$ be the set of vectors V in $\mathbb{R}^4$. Let $x^1,\ldots,x^4$ be the coordinates in $\mathbb{R}^4$. Then $x^1,\ldots,x^4$, $\dot{x}^1,\ldots,\dot{x}^4$ are coordinates in $T^1(\mathbb{R}^4)$ where $\dot{x}^i(V) = a^i$ for the vector

$$V = a^i \partial/\partial x^i$$

while $x^i(V)$ is the x^i coordinate of the base point of V.

A vector field in $T^1(\mathbb{R}^4)$ is generically of the form

$$I = a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial \dot{x}^i}$$

Such a vector is called <u>basic</u> if the component a^i is precisely $\dot{x}^i$.

Consider the cartesian product

$$T^1(\mathbb{R}^4) \times \ldots \times T^1(\mathbb{R}^4) \qquad (N \text{ factors}). \tag{2.1}$$

If $1 \leq n \leq N$ let a vector field on the space (2.1) of the form

$$I_n = \dot{x}^i_n \cdot \frac{\partial}{\partial x^i_n} + A^i_n \cdot \frac{\partial}{\partial \dot{x}^i_n} \tag{2.2}$$

be called n-basic. Here x^i_n and $\dot{x}^i_n$ refer to the x^i and $\dot{x}^i$ of the n-th factor in (2.1).

A (second order) N particle interaction is characterized by N vector fields $I_1,\ldots,I_N$ where I_n has the n-basic form (2.2) and the various fields commute:[1)]

$$[I_k, I_m] = 0 \qquad 1 \leq k, m \leq N \tag{2.3}$$

The interaction is Poincaré invariant if each I_n is invariant under the action of the Poincaré group in (2.1).

Let F_n be the n-basic vector field[2)]

$$F_n = u^k_n \cdot \frac{\partial}{\partial x^k_n} \quad . \tag{2.4}$$

Then $F_1,\ldots,F_N$ define the zero-interaction.

We want to show when such a map φ of (2.1) onto itself can be used to make a non-zero interaction.

Let

$$\varphi : \mathcal{P} \longrightarrow \mathcal{P} \tag{2.5}$$

be a smooth map of that part P of (2.1) on which all the u_n are timelike with $u^4_n > 0$ and (on which) the differences $x_m - x_n$ are spacelike

If we have a map φ as in (2.5), then $x^i_m(\varphi(p))$ is the new coordinate of p . We will denote it by X^i_m (so $X^i_m = x^i_m \circ \varphi$ in conventional notation). $u^i_m \circ \varphi$ we denote by U^i_m . Any set of four components such as $X^1_m,\ldots,X^4_m$ will be abbreviated by a single letter, here X_m . Vector fields for us are differential operators (implicit in (2.3), and explicit in (2.4)). When a vector F is applied to a function g , the result (of the differentiation) we denote by $F[g]$.

2.6 THEOREM. Suppose φ is 1:1 and maps P onto P , with a smooth inverse φ^{-1} . Then the image vector field I_n of F_n under φ will be m-basic if and only if

$$F_n[X_m] = 0 \quad \text{for} \quad m \neq n \tag{2.6.1}$$

$$F_n[U_m] = 0 \quad \text{for} \quad m \neq m \tag{2.6.2}$$

and

$$F_n(X_n) = U_n \tag{2.6.3}$$

If these conditions are met then the ensuing interaction will be non-zero if and only if

$$F_m[F_m[X_m]] \neq 0 \text{ for some } m \tag{2.6.4}$$

The interaction will be Poincaré invariant if φ commutes with the action of the Poincaré group in P.

It is easy to vizualize how a map φ changes one vector into another, and it is easy to see that if it is 1:1, then it changes one vector field into another. The formalism is that if φ changes vector field F to vector field I then

$$I[g] = F[g \circ \varphi] \circ \varphi^{-1}$$

for each differentiable function g defined on the manifold (say P). If I_n is to be n-basic, then

$$I_n = u_n^i \frac{\partial}{\partial x_n^i} + \text{terms of type } \frac{\partial}{\partial u}$$

Therefore $I_n[x_m] = 0$ for $m \neq n$. But $I_n[x_m] = F_n[x_m \circ \varphi] \circ \varphi^{-1} = F_n[x_m] \circ \varphi^{-1}$. Thus (2.6.1).(2.6.3) follows from $I_n[x_n] = u_n$. (2.6.2) follows from the fact that I_n should contain no $\partial/\partial u_m$ for $m \neq n$.

So now

$$I_n = u_n^i \frac{\partial}{\partial x_n^i} + A_n^i \frac{\partial}{\partial u_n^i}$$

If some A_n^i is not 0, the interaction is not zero. Now

$$A_n = I_n[u_n] = F_n[u_n \circ \varphi] \circ \varphi^{-1} = F_n[U_n] \circ \varphi^{-1} \tag{2.7}$$

Together with (2.6.3), this yields (2.6.4).

A generic set $(W_1, \ldots, W_N)$ of world "lines" for the interaction

$\varphi_*(F_1),\ldots,\varphi_*(F_N)$ is the image $\varphi(L_1,\ldots,L_N)$ of a generic set $L_1,\ldots,L_N$ of world lines for $F_1,\ldots,F_N$. If T is any Poincaré map, then

$$(T(W_1),\ldots,T(W_n)) = \varphi(L_1',\ldots,L_n')$$

where $L_m' = T(L_m)$. Thus our new interaction is Poincaré invariant. This ends the proof of (2.6).

A motion for the interaction $I_1,\ldots,I_N$ is an N-dimensional submanifold S to which each of the vector fields $I_1,\ldots,I_N$ is tangent. When this S is projected into $\mathbb{R}^4$ by x_n a curve is obtained: the world line of the n-th particle for that motion. If φ satisfies (2.6) then a motion S of $E_1,\ldots,E_N$ is transformed by φ into a motion for $I_1,\ldots,I_N$.

A typical motion for $F_1,\ldots,F_N$ is obtained by selecting a point $a_n \in \mathbb{R}^4$ and a timelike unit vector v_n in $\mathbb{R}^4$ for each $n = 1,\ldots,N$. Then as the real parameters $s_1,\ldots,s_N$ vary over $\mathbb{R}^N$, the point

$$(a_1+s_1v_1,\ \ldots,\ a_N+s_Nv_N)$$

sweeps out a motion for $F_1,\ldots,F_N$. The image under φ is a motion for $I_1,\ldots,I_N$ and the world line for the n-th particle is given in $\mathbb{R}^4$ by

$$x_n = X_n(a_1+s_1v_1,\ \ldots,\ a_N+s_Nv_N) \tag{2.8}$$

3. Potential examples

Let the pseudoeuclidean or Minkowski scalar product of a and b in $\mathbb{R}^4$ be denoted by (ab). Abbreviate $x_m - x_n$ by x_{mn}. Define

$$g_{mn} = \begin{vmatrix} (u_n u_n) & (u_n x_{mn}) \\ (u_n x_{mn}) & (x_{mn} x_{mn}) \end{vmatrix}$$

for $m,n = 1,2,\ldots,N$. In terms of (2.4) $F_p[x_{mn}] = 0$ if $p \neq m,n$, and $F_m[x_{mn}] = u_m$. So

$$F_n[g_{mp}] = 0 \quad \text{if} \quad m \neq n \tag{3.2}$$

and

$$F_n[g_{np}] = \varepsilon(u_p u_p)\cdot(u_n x_{np}) - \varepsilon(u_n u_p)\cdot(u_p x_{np}) \tag{3.3}$$

If we let, for all $m = 1,2,\ldots,N$,

$$X_m = x_m + \sum_p f_{mp} u_p \tag{3.4}$$

where f_{mp} depends only on the g_{mn} (m fixed)and define

$$U_m = F_m[X_m]$$

we obtain a φ which commutes with the Poincaré group and also satisfies all conditions of (2.6) except for the existence of an inverse on P .

Assume therefore that it has an inverse.

Let us examine the resulting interaction.

By (2.9) and (3.2) the world lines are given by

$$x_m = a_m + s_m v_m + \sum_n f_{mn} v_n \tag{3.5}$$

where the $g_{k\ell}$ "inside" f_{mn} have also to be expressed in terms of the v_i and $a_j + s_j v_j$.

The theory says that (3.5) should describe a curve. This can be verified easily, as follows. For simplicity, let $N = 2$. Let $x_1 = a_1 + v_1 s_1$, $x_2 = a_2 + v_2 s_2$. This leads to

$$g_{12} = (aa) + 2(av_1)s + s^2 - [(av_2) + (v_1 v_2)s]^2 \tag{3.6}$$

where $a = a_1 - a_2$ and $s = s_1$. The s_2 evaporates.

Equation (3.6) shows that for $v_1 = v_2$, the g_{12} is constant so that (3.4) is a straight line. When $v_1 \neq v_2$ then $g_{12} \to -\infty$.

One could therefore define f_{12} in such a way that $f_{12} \to 0$ as $g_{12} \to -\infty$. Then it would follow that the world line (3.5) always becomes more and more straight as $s\ (=s_1)$ tends to $\pm\infty$.

4. The case of two particles

For $N = 2$ we can pretty well say how X_1 and X_2 should be chosen to fulfill condition (2.6.1.) . Certainly we are not limited to the form (3.6) .

Before proceeding, let us abbreviate $x_1, x_2, u_1, u_2, X_1, X_2, U_1, U_2$, x_{12} and X_{12} by $x, y, u, v, X, Y, U, V, z$ and Z respectively.

When $N = 2$ we are justified, by the assumption of Poincaré invariance, to assume space-time dimension $2+1$.

$X - x$ is evidently translation invariant, and transforms like a vector under the Lorentz group. In 3-dimensional space-time, u, v and z together form a basis for such vectors (save for unimportant exceptions). Hence

$$X = x + \alpha z + \beta u + \gamma v \tag{4.1}$$

where these Greek letters depend only on the Lorentz invariants of u, v, z.

The following six functions are Lorentz-invariant: (uu), (uv), (vv), $G_{12} = 2(vv)(uz) - 2(uv)(vz)$, $G_{21} = -2(uu)(vz) + 2(uv)(uz)$, and

$$\tau^2 = \begin{vmatrix} (uu) & (uv) & (uz) \\ (vu) & (vv) & (vz) \\ (zu) & (zv) & (zz) \end{vmatrix}$$

Any function α of these is also invariant. Conversely, if α is a Lorentz-invariant, then, on that set where

$$G = (uu)(vv) - (uv)^2 \tag{4.1.1}$$

is not 0, α can be expressed in terms of these six functions.

Let $\mathcal{U}_i (i = 1,2)$ be the class of those invariant functions α such that $F_i[\alpha] = 0$. With the aid of the facts $F_2[u] = F_2[v] = 0$, $F_2[z] = -v$, $F_2[(uz)] = -(uv)$, etc., one can easily see which invariant functions are in $\mathcal{U}_2$.

4.2. PROPOSITION. $\mathcal{U}_2$ *consists of those functions expressible in terms of* (uu), (uv), (v,v), τ^2, *and* G_{12}.

We now ask what (2.6.1) requires of X as in (4.1).

4.3. PROPOSITION. $F_2[X] = 0$ *precisely if* α, β *and* $\gamma + \frac{(vz)}{(vv)}\alpha$ *belong to* $\mathcal{U}_2$. *Put another way,* $F_2[X] = 0$ *if and only if*

$$X = x + \alpha z + \beta u + \left(\delta - \frac{(vz)}{(vv)}\alpha\right) v$$

where α, β, δ *are chosen from* $\mathcal{U}_2$.

Passing over the trivial proof of this, we enquire what (2.6.3) says about U, which (we recall) is $u \circ \varphi$.

Proof. Suppose u and v have the same value b at some point of P (see (2.5)) . Select a point a of $\mathbb{R}^4$ such that $(ab) = 0$, where a is spacelike. Consider the point p_a of P where $u = v = b$ and $x = -y = a$. The value of (4.6) is here the same as at the former point. Suppose that value is $-\frac{1}{2}$. A simple computation shows

$$X = \beta b + \delta b \quad , \quad Y = \bar{\beta} b + \bar{\delta} b$$

where the arguments in $\beta, \ldots, \bar{\delta}$ are as in (4.6), whence independent of a . Further, $u = v$ makes $G = 0$ and as a matter of fact, it makes $U = b$, $V = b$ (compare, for example, (4.5)). Thus the point $\varphi(p_a)$ to which φ maps p_a is independent of a . This violates the crucial assumption that φ is 1:1 .

4.8. LEMMA. Let Q be that part of P (see (2.5)) on which $u=v$ and $(uz) = (vz) = 0$. Let φ be a map of $P \to P$ as above defining an interaction of substitution type. Then φ maps Q onto itself, as does φ^{-1} .

Proof. If $u = v$ and $(vz) = 0$ then $X = x(\varphi) = x + \alpha z + \beta u + \delta u$ and $Y = y - \bar{\alpha} z + \bar{\beta} u + \bar{\delta} u$. Hence, using (4.6), $Z =$ the value of z at the image point is $(1 + \alpha + \bar{\alpha})z$. We must calculate also U and V (the values of u and v at the image point). Certainly $G = 0$, so

$$U = u + \alpha u + (-\alpha) u = u$$

and similarly

$$V = v$$

Thus $(UZ) = (VZ) = 0$ and $V = U$. Hence the image point is still in Q .

Now let q be a point of Q where $u = v = c$, $x = a$, $y = b$, whence $((b - a)c) = 0$. Let p be the point where $(x-y)(1+\alpha_0+\bar{\alpha}_0) = a - b$, $2(\beta_0 + \delta_0)c + x + y = a + b$, $u = v = c$, whence α_0 and $\bar{\alpha}_0$ means that α and $\bar{\alpha}$ are evaluated as in (4.6) for $u = c$. This p is surely a point of Q and one can easily verify that $\varphi(p)$ is the given q .

4.9. THEOREM. Let S be a motion for a symmetric, Poincaré- invariant 2-particle interaction of substitution type. Suppose in some Lorentz frame, the two particles executing the motion S are at rest for $t = 0$. Then in that frame they have been and will be forever at rest.

Proof. Let the Lorentz frame in which they appear to be at rest

for $t = 0$ correspond to the cartesian coordinates. Let the positions be $(a_1,a_2,a_3,0) = a$ and $(b_1,b_2,b_3,0) = b$. We may suppose that u and $v = (0,0,0,1) = c$. Then $(a,b,c,c) = q$ is a point of Q. By (4.8) there is a p in Q such that $\varphi(p) = q$. Find the motion S_0 relative to the dynamics F_1,F_2 containing the initial p. Then $S = \varphi(S_0)$ will be the motion relative to the dynamics I_1, I_2 containing q. By (4.8), this motion S lies in Q.

We will now show that $I_n = F_n$ on Q. The form of I_n is given in (2.2) where the A_n are given by (2.7). We consult (4.4) for $U_1 = U$ and apply F_1. We again observe that $F_1[\alpha] = 2G\alpha'$, so that $F_1[\alpha'] = 2G\alpha''$, etc. Now $A_1 = F_1[U]\circ\varphi^{-1}$. After simplification we obtain

$$A_1 = 4\left\{G^2\alpha''z + G\alpha'u + G^2\beta''u + G^2\left(\delta'' + \frac{(vz)}{(vv)}\alpha''\right)v\right\}\circ\varphi^{-1}$$

Since φ^{-1} maps Q onto Q, and G (see (4.1)) is 0 on Q, we have A_1 (and also A_2) being 0 on Q. Therefore $I_n = F_n$ on Q. Hence the motion S appropriate for I_1, I_2 is appropriate for F_1, F_2 i.e. it is a motion for the zero interaction. Thus on S the velocities of the particles are constant, as (4.9) asserts.

FOOTNOTES

1) This was first explicitely stated by Ph. Droz-Vincent, Relativistic systems of Interacting particles, Physica Scripta, Vol. 2, 129-134 (1970).

2) Henceforth we abbreviate $\dot{x}_n^k$ by u_n^k.

3) See R. Arens, An Analysis of Relativistic Two-Particle Interactions, Arch. Rat. Mech. Anal., Vol. 47, 255-271 (1972)

TRANSLATION, DILATION, LORENTZ INVARIANT TWO-PARTICLE INTERACTIONS

Richard Arens

Mathematics Department
University of California, Los Angeles
Los Angeles, CA 90024

We show the existence of an infinite number of symmetric 2-particle interactions. These interactions are translation invariant, Lorentz invariant, and lead to second order differential equations. Our examples are analytic.

1. Introduction

The interactions we construct are local in the sense that the power series involved do not necessarily converge for all relevant values of the initial conditions.

They do, however, converge at those initial conditions where both particles start from rest in some frame. These most crucial initial conditions could not be satisfied by the earliest example of a local interaction, the one due to Kerner [1, pp. 262-263] .

"Symmetric" means that if the initial conditions of the particles are interchanged, then the same two world lines will still be the ensuing motion. A global symmetric interaction was given in [2,7.5] as well as earlier, by Arens and Babbitt [see 1] but these were not analytic (merely C^∞).

We were led to these interactions by reducing the arbitrariness of the interactions by improving an extra condition: dilation invariance. That is to say, if the pair of world lines W_1, W_2 belongs to the interaction and T is an expansion of space time centered on the origin $(0,0,0,0)$, then TW_1, TW_2 also belongs to the interaction.

There was a danger that this extra requirement would force the interaction to reduce to the zero interaction, but the reverse is

true. There are such interactions, locally, at least.

The resulting differential equations are simpler than for the general Poincaré-invariant case. Therefore the chance of someone finding an explicit solution is enhanced.

2. The formalism of dilation-invariant interactions

Any binary interaction can be described by giving the Minkowski accelerations $M_i (i = 1,2)$ for the two world lines involved, and these accelerations have the form

$$M_i = f_i [z_i - (u_i \cdot z_i) u_i] + g_i [u_i - (u_1 \cdot u_2) u_i] \qquad (2.1)$$

where u_1, u_2 are the space time velocities (with $u_i \cdot u_i = 1$, the dot product with signs $-,-,-,+$ being intended throughout) while z_i is the position of the i-th particle relative to the other one (the i'-th). Here $f_1, \ldots, g_2$ are four scalar functions of the positions and the (unit) velocities of the two particles [2,2.4].

An interaction is invariant under a space-time transformation T if whenever (W_1, W_2) is a pair of world lines associated by the interaction, then so are TW_1 and TW_2 [1]. The zero interaction is invariant under each dilation (defined by some $\rho > 0$ and sending each x into ρx). Kepler's third law shows that Newton's inverse square law is not dilation invariant.

However, in a relativistic context, dilation invariance does not seem to be an unnatural requirement. It can be easily characterized.

Proposition 2.2. The interaction described by 2.1 is dilation invariant precisely if for each positive number ρ,

$$f_i (\rho x_1, \rho x_2, u_1, u_2) = \rho^{-2} \cdot f_i (x_1, x_2, u_1, u_2)$$

$$g_i (\rho x_1, \rho x_2, u_1, u_2) = \rho^{-1} \cdot g_i (x_1, x_2, u_1, u_2), \qquad i = 1,2$$

For the translation-and-Lorentz (= Poincaré) invariant interaction, the functions $f_1, \ldots, g_2$ depend only on the four Minkowski invariants of the three vectors $u_1, u_2, z = x_1 - x_2$ [2, (2.2)]. If we add to this dilation invariance, then they depend in an essential way only on three. We recall the notation of [2, sec. 5]: $\lambda = u \cdot z$, $\mu = v \cdot z$, $\nu = u \cdot v$, $\zeta = z \cdot z$, $f = f_1$, $g = g_1$, $h = f_2$, $k = g_2$.

Proposition 2.3. Let the interaction 2.1 be dilation and Poincaré invariant. Then there are functions F, G, H, K such that

$$f = \frac{1}{-\zeta} F\left(\frac{\lambda}{\sqrt{-\zeta}}, \frac{\mu}{\sqrt{-\zeta}}, \nu\right)$$

$$g = \frac{1}{\sqrt{-\zeta}} G\left(\frac{\lambda}{\sqrt{-\zeta}}, \frac{\mu}{\sqrt{-\zeta}}, \nu\right)$$

$$h = \frac{1}{-\zeta} H\left(\frac{\lambda}{\sqrt{-\zeta}}, \frac{\mu}{\sqrt{-\zeta}}, \nu\right)$$

$$k = \frac{1}{\sqrt{-\zeta}} K\left(\frac{\lambda}{\sqrt{-\zeta}}, \frac{\mu}{\sqrt{-\zeta}}, \nu\right)$$

3. The fundamental relation

Let L_1 be the differential operator defined for functions of u_1, u_2, x_1, x_2 such that

$$L_1(x_1) = u_1 \,, \quad L_1(u_1) = M_1 \,, \quad L_1(x_2) = 0 \,, \quad L_1(u_2) = 0$$

Similarly let L_2 be defined by

$$L_2(x_1) = 0 \,, \quad L_2(u_1) = 0 \,, \quad L_2(x_2) = u_2 \,, \quad L_2(u_2) = M_2$$

The fundamental relation which any interaction must satisfy is [2,4.5]

$$L_1(M_2) = 0 \,, \qquad L_2(M_1) = 0 \tag{3.1}$$

The intent of an equation like this is that M_2 should be represented through its Cartesian components and that L_1 of each component should vanish. This requires of f,g,h,k (or of F,G,H,K) that they should satisfy some differential equations which we calculate below.

Such differential equations were presented in [2,4.8] . They were in terms of the variables λ, μ, ν, ζ and so apply only in such regions (or for such initial conditions) where these variables form differentiable coordinates. Now ν is not an acceptable coordinate at the very important initial condition

$$u_1 = (0,0,0,1) \,, \quad u_2 = (0,0,0,1) \,, \quad x = (b,0,0,0) \tag{3.2}$$

At (3.2) , the variable ν assumes the value 1 and since $\nu \geq 1$ in general, it cannot be a coordinate there. Since we are interested in local solutions to the fundamental relation valid at, and near, (3.2) we must choose a variable τ which can replace ν and serve as

a coordinate.

Such a variable τ is the <u>determinant of</u> u_1 , u_2 , z (please note that for valid reasons, our space-time is 3-dimensional [2, 2.1]).

We augment the list $L_1(\lambda) = 1 + f(\zeta - \lambda^2) + g(\mu - \lambda\nu)$, $L_1(\mu) = \nu$, $L_1(\nu) = f(\mu - \lambda\nu) + g(1 - \nu^2)$, $L_1(\zeta) = 2\lambda$ given in [2,4.6] by

<u>Proposition 3.3</u>. $L_1(\tau) = -(\lambda f + \nu g)\tau$.

Proof. $L_1(\det(u_1,u_2,z)) = \det(M_1,u_2,z) + \det(u_1,0,z) + \det(u_1,u_2,u_1)$. From this and (2.1) we obtain (3.2).

We can now write down the differential operator L_1 , using the coordinates λ,μ,ζ,τ in terms of which any Poincaré invariant function φ may be expressed. We have the components $L_1(\tau)$, etc., and so

$$L_1 = [1 + (\zeta - \lambda^2)f + (\mu - \lambda\nu)g]\cdot\frac{\partial}{\partial\lambda} + \nu\cdot\frac{\partial}{\partial\mu} - \tau(\lambda f + \nu g)\frac{\partial}{\partial\tau} + 2\lambda\frac{\partial}{\partial\zeta} \tag{3.4}$$

Here ν is not a coordinate. It is, of course, expressible in terms of λ,μ,ζ,τ .

The exact relation is

$$\nu = \frac{\lambda\mu}{\zeta} + \left[\frac{(\lambda^2-\zeta)(\mu^2-\zeta)}{\zeta^2} - \frac{\tau^2}{\zeta}\right]^{1/2} \tag{3.5}$$

Since ζ is negative, one can easily deduce that $\nu \geq 1$, and that $\nu = 1$ only if $\lambda=\mu$ and $\tau = 0$. Thus the differential operators L_1, L_2 can not be used in the form written in [2] for considerations at and near the conditions (3.2).

The derivation of (3.5) may be left to the reader.

The expression on the right of (3.5) is analytic in a neighborhood of any point of its domain, because, as is natural, we require a space-like separation of z_1,z_2 which makes $\zeta < 0$.

The fundamental equations are (cf. 2,5.3)

$$L_1(h) = fk \quad , \quad L_1(k) = h + (\nu g + \lambda f)k$$
$$L_2(f) = gh \quad , \quad L_2(g) = f + (\nu k - \mu h)g \tag{3.6}$$

where L_1 is given by (3.4). L_2 is given by

$$L_2 = -[1 + (\zeta - \mu^2)h - (\lambda - \mu\nu)k]\frac{\partial}{\partial\mu} - \nu\frac{\partial}{\partial\lambda} - \tau(-\mu h + \nu k)\frac{\partial}{\partial\tau} - \tau\mu\frac{\partial}{\partial\zeta} \tag{3.7}$$

One obtains (3.7) from (3.6) by making those changes required for interchanging the particles, namely

$$(\lambda, \mu, \zeta, \tau) \longrightarrow (-\mu, -\lambda, \zeta, \tau)$$

4. The fundamental relations if dilation-invariance is also imposed

We introduce coordinates $x = \lambda(-\zeta)^{-1/2}$, $y = \mu(-\zeta)^{-1/2}$, $\sigma = \tau(-\zeta)^{-1/2}$. This makes

$$\nu = -xy + [(1+x^2)(1+y^2) + \sigma^2]^{1/2} \tag{4.1}$$

Therefore, the functions F,G,H,K of (2.3) may be presumed to be functions of x,y,σ. We compute the x-component $L_1(x)$ of L_1, etc., and assemble the differentiable operator L_1 in terms of the new variables. The computations produce

$$L_1 = (-\zeta)^{-1/2}\left(\Lambda_1 - 2x\zeta\frac{\partial}{\partial\zeta}\right) \tag{4.2}$$

where Λ_1 is the differential operator

$$[1 + x^2 - (1+x^2)F + (y - xy)G]\frac{\partial}{\partial x} + (\nu + xy)\frac{\partial}{\partial y} + (x - xF - \nu G)\sigma\frac{\partial}{\partial\sigma} \tag{4.3}$$

The companion operator L_2 is

$$L_2 = (-\zeta)^{-1/2}\left(\Lambda_2 + 2y\zeta\frac{\partial}{\partial\zeta}\right) \tag{4.3a}$$

where

$$\Lambda_2 = -[1 + y^2 - (1+y^2)H - (x - y\nu)K]\frac{\partial}{\partial y} - (\nu + xy)\frac{\partial}{\partial x} - (y - yH + \nu K)\sigma\frac{\partial}{\partial\sigma} \tag{4.4}$$

Theorem 4.5. The four functions of (2.3) define an interaction if and only if they satisfy the fundamental equations

$$\Lambda_1(H) = -2xH + FK$$
$$\Lambda_1(K) = -xK + H + (\nu G + xF)K$$

4.4. PROPOSITION

$$U = u + 2G\alpha' z + \alpha u + 2G\beta' u + \left(2G\delta' - \frac{(vu)}{(vv)}\alpha + \frac{(vz)}{(vv)} 2G\alpha'\right) v$$

where G is given by (4.1.1) while α' (etc.) means $\partial\alpha/\partial G_{12}$ presuming that α is expressed in terms of the five functions named in (4.3).

The proof is immediate in view of $F_1[\alpha] = \alpha' 2G$, etc.

There is a similar expression for $V = U_2$:

$$V = v - 2G\bar{\alpha}' z + \bar{\alpha} v + 2G\bar{\beta}' v + \left(2G\bar{\delta}' - \frac{(uv)}{(uu)}\bar{\alpha} - \frac{(uz)}{(uu)} 2G\bar{\alpha}'\right) u \qquad (4.5)$$

obtained from applying F_1 to

$$Y = y - \bar{\alpha} z + \bar{\beta} v + \left(\bar{\delta} + \frac{(uz)}{(uu)}\bar{\alpha}\right) u$$

Here the bars are used only to indicate corresponding terms. Certain signs are changed because $z = x - y$ changes sign when x and y are interchanged. $\bar{\alpha}'$ in (4.5) indicates $\partial\bar{\alpha}/\partial G_{21}$.

In a symmetric[3] interaction, one would have

$$\alpha((uu),(uv),(vv),\tau^2,G_{12}) = \bar{\alpha}((vv),(uv),(uu),\tau^2,G_{21})$$

(also for β and δ).

For a symmetric interaction one has

$$\alpha = \alpha((uu),(uu),(uu),0,0) = \bar{\alpha} \qquad (4.6)$$

when $v = u$ (and the same for β, δ).

We want to prove that when $u = v$, the forces of interaction vanish, although the forces need not be 0 when $u \neq v$. To form our hypothesis more precisely let us say that a Poincaré-invariant 2-particle interaction is of substitution type exactly when there is a 1:1 map φ (as in (2.6)) for which

$$I_i = \varphi_*(F_i) \quad , \qquad i = 1,2$$

We establish a detail which is needed later.

4.7. PROPOSITION. Consider a symmetric interaction of substitution type. Then the value (4.6) assumed by α and $\bar{\alpha}$ for $u = v$ cannot be $-\frac{1}{2}$.

$$\Lambda_2(F) = 2yF + HG$$
$$\Lambda_2(G) = yG + F + (\nu K - yH)G$$

The proof consists in writing the relations (3.6) using the formulas (4.3), (4.4) and those of (2.3). Certain powers of $-\zeta$ can then be factored out, giving the equations above. The functions $F,\dots,K$ depend only on x,y,σ.

We showed in [2] that any solution in which G and K are identically 0 must have F and H equal 0 also, thus representing the zero-integration.

Solutions in which F and H are 0 are called <u>purely kinetic</u> [2]. We write down the equations for such interactions:

$$[1+x^2+(y-x\nu)G]\cdot\frac{\partial K}{\partial x} + (\nu+xy)\frac{\partial K}{\partial y} + (x-\nu G)\sigma\cdot\frac{\partial K}{\partial \sigma} = \\ = -xK + \nu GK \tag{4.6}$$

and

$$-(\nu+xy)\frac{\partial G}{\partial x} - [1+y^2-(x-y\nu)K]\frac{\partial G}{\partial y} - (y+\nu K)\sigma\cdot\frac{\partial G}{\partial \sigma} = \\ = yG + \nu KG \tag{4.7}$$

The other two equations reduce to $0 = 0$.

We shall see that there are <u>power series</u> solutions of these equations, and therefore, of course, of all the preceding, more general, equations.

As an appendix to this section, we record for later reference the form of the operators Λ_1 and Λ_2 in terms of x,y,ν:

$$\Lambda_1 = [1+x^2-(1+x^2)F+(y-x\nu)G]\frac{\partial}{\partial x} + (\nu+xy)\frac{\partial}{\partial y} + \\ + [(y-\nu x)F + (1-\nu^2)G]\frac{\partial}{\partial \nu} \tag{4.8}$$

$$\Lambda_2 = -(\nu+xy)\frac{\partial}{\partial x} - [1+y^2-(1+y^2)H-(x-y\nu)K]\frac{\partial}{\partial y} + \\ + [(\nu y-x)H + (1-\nu^2)K]\frac{\partial}{\partial \nu} \tag{4.9}$$

5. <u>Symmetric interaction</u>

For a generic function J of the four (vector) variables u_1,u_2, z_1,z_2 appearing in (2.1), define

$$J^*(u_1,u_2,z_1,z_2) = J(u_2,u_1,z_2,z_1)\ .$$

If J depends only on the variables x,y,σ,ζ mentioned in the last section, then

$$J^*(x,y,\sigma,\zeta) = J(-y,-x,-\sigma,\zeta)$$

An interaction of two particles is <u>symmetric</u> [2,7] if $H = F^*$ and $K = G^*$. For a symmetric interaction there are only two functions needed, say F and G. They have to satisfy two equations of (4.5):

$$\Lambda(F) = 2yF + F^*G \tag{5.1}$$

$$\Lambda(G) = yG + F + (\nu G^* - yF^*)G \tag{5.2}$$

Λ being the following modification of Λ_2:

$$\begin{aligned}\Lambda = &-[1+y^2-(1+y^2)F^*-(x-y\nu)G^*]\frac{\partial}{\partial y} - \\ &-(\nu+xy)\frac{\partial}{\partial x} - (y-yF^*+\nu G^*)\sigma\frac{\partial}{\partial\sigma}\end{aligned} \tag{5.3}$$

If we are content with a purely kinetic interaction, we may take $F = 0$ and are left with only one equation

$$\begin{aligned}&[1+y^2-(x-y\nu)G^*]\frac{\partial G}{\partial y} + (\nu+xy)\frac{\partial G}{\partial x} + \\ &+(y+\nu G^*)\sigma\frac{\partial G}{\partial\sigma} = -yG - \nu GG^*\end{aligned} \tag{5.4}$$

Here ν is as always given by (4.1). It is evidently an analytic function of x,y,σ in a neighborhood of the point $(0,0,0)$ corresponding to the initial conditions (3.2).

<u>Theorem 5.5. There is a local binary interaction valid in a neighborhood of the initial conditions (3.2) which is</u>

<u>analytic</u> ,	(5.5.1)
<u>symmetric</u> ,	(5.5.2)
<u>Poincaré and dilation invariant</u> ,	(5.5.3)
<u>purely kinetic</u> ,	(5.5.4)
<u>non-zero</u> .	(5.5.5)

The only blemish in this theorem is that the acceleration function it delivers is perhaps not defined for all initial velocities and relative positions. This is what we mean by local.

Our solution will depend on a variant of the Cauchy-Kowalewski existence theorem [3 , vol.5] . It may be that for some initial Cauchy data the solution will indeed converge everywhere but we can promise only a local solution.

If it were not for those asterisks in (5.4) , one could quote the Cauchy-Kowalewski theorem and assert that, for any analytic boundary value function

$$\Gamma(x,\sigma)$$

one can find an analytic $G(x,y,\sigma)$ defined at and near $(0,0,0)$, satisfying (5.4) . This would be because the coefficient of $\partial G/\partial y$ in (5.4) is 1 and therefore not 0 , at the point $(0,0,0)$.

However, in our differential equation there is present the "unknown" function G and its "conjugate" G^* . We will show how to treat this problem. We first change to new variables s,t,σ where $s = x+y$, $t = x - y$. This changes (5.4) to

$$\begin{aligned}&[1+y^2-(x-yv)G^*+v+xy]\frac{\partial G}{\partial s} =\\ &=-[1+y^2-(x-yv)G^*-v-xy]\frac{\partial G}{\partial t} -\\ &-(y+vG^*)\sigma\frac{\partial G}{\partial\sigma}-yG-vGG^*\end{aligned} \tag{5.6}$$

Here x has to be thought of as $\frac{1}{2}(s+t)$ and y as $\frac{1}{2}(s-t)$. The nature of the conjugation has changed, in that $G^*(s,t,\sigma) = G(-s,t,\sigma)$. Dividing by the coefficient of $\partial G/\partial s$, we obtain a differential equation of the following kind:

$$\frac{\partial G}{\partial s} = \Phi(s,G,G^*,\ldots) \tag{5.7}$$

Where the dots stand for other variables $t_1,\ldots,t_n$, and the derivatives of G and G^* with respect to $t_1,\ldots,t_n$. Here Φ is an analytic function of all its arguments, and $G^*(s,\ldots) = G(-s,\ldots)$, the variables represented by dots being unaffected.

Proposition 5.8. Let a function $\Gamma(t_1,\ldots,t_n)$ be a function depending analytically on its arguments. Then there is a solution of (5.7) with $G(0,t_1,\ldots) = \Gamma(t_1,\ldots)$.

Instead of proving this in the conventional way, I will just do an example illustrating how Problem (5.7) is reduced to a Cauchy problem of the usual sort. Let the equation to be solved be

$$G_1(s,t) = (s+t)\cdot G(s,t) + G(s,t)\cdot G(-s,t) \qquad (5.9)$$

Here G_1 indicates $\partial G/\partial s$. Let $G(s,t) = A(s,t) + B(s,t)$ where A is an even, and B an odd, function of s , respectively. Then

$$A_1+B_1 = (s+t)(A+B) + (A+B)(A-B)$$

We replace s by $-s$ and obtain

$$-A_1+B_1 = (-s+t)(A-B) + (A-B)(A+B)$$

because the derivative A_1 is odd, and B_1 is even. These two equations are equivalent to the system

$$A_1 = sA + tB \qquad (5.9.1)$$

$$B_1 = tA + sB + A^2 - B^2 \qquad (5.9.2)$$

with the initial conditions

$$A(0,t) = \Gamma(t) \quad , \quad B(0,t) = 0 \qquad (5.9.3)$$

This can be solved by the usual Cauchy-Kowalewski theorem. Let the solution yield $A(s,t)$, $B(s,t)$. The question arises: is A an even, and B an odd, function of s ? To see this, let $\xi(x,t) = A(-s,t)$, $\eta(s,t) = -B(-s,t)$. It is easily verified that the pair ξ,η satisfies (5.9.1), (5.9.2), (5.9.3). Thus $\xi = A$ and $\eta = B$. Letting $G=A+B$ solves (5.9).

6. Pseudo-symmetric interactions

In such an interaction, $f = h$ and $g = k$, by definition. The concept was introduced in [2] because the differential equations then take on a more classical form. Indeed, we proved in [2] that each such an interaction could be obtained by imposing a relation on three expressions, namely [2, 6.4.5, 6.4.6, 6.4.7] .

Theorem 6.1. Any Poincaré and dilation invariant, analytic pseudo-symmetric interaction which is valid at the initial conditions (3.2) is the zero-interaction.

Proof. It is known [2,6.4] that such an interaction involves an

analytic relation between three integrals [2, 6.4.5, 6.4.6, 6.4.7] . In terms of ζ and the new variables x,y,σ , these integrals may be taken as

$$(1-\nu^2)G^2 + 2(x-y)(\nu+1)G - 2\nu + 2 \tag{6.1.1}$$

$$G\sigma \tag{6.1.2}$$

$$G^2(-2xy\nu + x^2 + y^2 + \sigma^2)\zeta^{-1} \tag{6.1.3}$$

where $G = g(-\zeta)^{1/2}$, the g being, as always, the g_1 of (2.1).

Let these integrals be denoted by φ,ψ,ω respectively. If the interaction is dilation invariant, the relation between φ,ψ,ω cannot involve ω since the ζ is not dilation invariant. Let the relation be

$$F(\varphi,\psi) = 0 \tag{6.2}$$

Since φ and ψ are 0 at (3.2), F has a vanishing constant term.

Proof. First let us assume that

$$F(z,w) = zA(z,w) + wB(z,w) \tag{6.3}$$

where

$$B(0,w) \neq 0 \qquad \text{and} \qquad A(z,0) \neq 0 \tag{6.4}$$

Now let $\sigma = 0$. Then G is still an analytic function of x and y . Note that $\psi = 0$ when $\sigma = 0$. Hence, for $\sigma = 0$ we obtain

$$0 = \varphi \cdot A(\varphi,0)$$

a non-trivial relation. It has the form

$$\varphi^m(a_0 + a_1\varphi + a_2\varphi^2 + \dots) = 0 , \qquad a_0 \neq 0 \tag{6.5}$$

No matter how G depends on x and y we can make φ (see (6.11)) as small as we like by making x close to y . Making φ small forces it to be 0 , by (6.5). But if $\varphi = 0$ then G certainly cannot be analytic, by the formula for solving $\varphi = 0$, even for $\sigma = 0$.

So F cannot look like (6.3) with (6.4) holding. This is to say, either

$$F = z^m \cdot G(z,w) \quad , \qquad m \geq 1$$

or

$$F = w^n D(z,w) \quad , \qquad n \geq 1 \qquad (6.6)$$

where

$$G(0,w) \neq 0 \quad , \qquad D(z,0) \neq 0$$

Now φ cannot be 0 as we have already noted, so $G(z,w) = 0$ is as good as $F = 0$. This puts us back into (6.3)-(6.4). This leaves case (6.6), so $\psi = 0$ (zero interaction) or we again fall into (6.3)-(6.4).

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SPONTANEOUS PREDICTIVISATION

L. Bel

Equipe de Recherche Associée au C.N.R.S. n° 533
Institut Henri Poincaré - 11, rue Pierre et Marie Curie
75005 PARIS, FRANCE

ABSTRACT

In the first section of this paper we define the concept of an Attractor of a hereditary first order differential equation as an ordinary differential equation whose solutions are solutions of the hereditary one and can be interpreted as the asymptotes of its generic solutions. We define also the concept of Predictive Differential Equations associated with a class of hereditary ones depending on a coupling constant G as a first order differential equation which is such that all its solutions are solutions of the corresponding hereditary one and which is analytic in the neighbourhood of $G = 0$. We report some numerical work proving that for some hereditary equations the corresponding predictive ones are Attractors.

In the second section we consider the retarded electromagnetic equations of two point charges and we prove numerically in a particular case that the associate Predictive Poincaré Invariant System defined in previous papers is an Attractor in an obviously generalized sense. Roughly speaking this means that the retarded electromagnetic equations of motion have a built-in mechanism which causes a spontaneous predictivisation of the causal interaction.

INTRODUCTION

This paper is divided in two sections. The first one has been included mainly to illustrate the concepts to be used in the second section, but the material that it contains could eventually be interesting for other purposes.

We consider a hereditary differential equation of the first order and we remember some fundamentals of the theory of such equations as the integration method of steps, and the regularization in the future of the solutions of the pure retarded equations.

Calling a Reduction of order J an ordinary differential equation which is such that its solutions are also solutions of the hereditary one we define an Attractor as a Reduction of order J which is such that its solutions can be interpreted in an appropriate sense, as asymptotes of the generic solutions of the hereditary equation.

We consider a class of first order hereditary equations depending on a coupling constant G and we define the associate Predictive Differential Equation as a Reduction of order 1 which is analytic in G in the neighbourhood of $G = 0$. A perturbation scheme is proposed to construct it.

Finally, for some hereditary equations, we prove by a numerical integration following the method of steps that their associate Predictive Differential Equations are Attractors. This result means that there is a collapse of the infinite dimensional configuration space to a finite dimensional one. We refer to this collapse as the spontaneous predictivisation of the hereditary equation. The word Predictive is justified in the sense that sometime after the beginning of the integration the Predictive Differential Equation and the value of the function at a given time permit to Predict to a good approximation the future of the solution.

Obviously the concepts and methods of this first section can be easily generalized to equations of higher order.

In the second section we consider the hereditary (causal) equation of motion of two point charges. When the radiation reaction forces are ignored the corresponding dynamical system is of second order but strictly speaking it is of the neutral type and therefore no regularisation in the future of the solutions obtained by the method of steps is to be expected. Actually we prove that at some level of approximation, and in two different meanings, there is such a regularization.

The consideration of the radiation reaction forces introduces some complication which we eliminate using the Order Reduction method. This

method reduces the third order hereditary dynamical system to a second order one which at the order of approximation that we consider is pure retarded (non neutral).

It is known how to associate to such a dynamical system a Predicitive Poincaré Invariant System (P.I.S.). We remember the usual procedure but we present also a constructive method which is closer to the concept of P.D.E. which we have presented in the first section and which is also better adapted to some numerical calculations.

We have integrated numerically following the method of steps, the equations of motion of two equal mass charges having equal absolute values for a fairly large class of initial conditions leading to planar motions and we have seen that the P.I.S. is an Attractor in the sense that the relative difference between the retarded acceleration and the predictive one tends to zero beyond a certain time as long as the distance between the two charges remains larger than a few natural units.

It was known that the P.I.S.'s associated with causal interactions permitted to deal with the finite dimensional space of solutions having the maximum smoothness. The numerical result reported in this section (we have proved a similar result for the gravitational interaction) seems to indicate that the theory of P.I.S.'s associated with causal interactions can be considered as an approximation which some time after the particles have been let free to interact becomes better and better as time elapses. To use an image drawn from thermodynamics we could say that the solutions of P.I.S.'s are the states of maximum entropy of the corresponding hereditary system and that retardation is the mechanism which pushes the entropy to increase steadily.

This paper is meant to discuss some ideas at a qualitative level and not at a rigorous one.

SECTION I : HEREDITARY DIFFERENTIAL EQUATIONS

1. Generalities [1] [2] [3]

Let us consider the following hereditary first order differential equation with one unknown x and one independent variable t which we call time:

$$\dot{x}_t = F(t, x_t ; x_{t-r}, \dot{x}_{t-r}) \tag{1.1}$$

where x_t and $\dot{x}_t$ mean the function $x(t)$ and its derivative at the time t and where x_{t-r} and $\dot{x}_{t-r}$ mean the same quantities at the time $t-r$, r being a positive constant, and F a conveniently smooth function of its arguments. Equation (1.1) is of course not the most general equation we can think of, but this type of equations will be sufficient to illustrate some points.

The initial value problem corresponding to equation (1.1) can be formulated as follows: how much do we have to know of the function $x(t)$ for values of t less or equal to t_o such that this information together with eq. (1.1) determine one and only one solution of class C^o for t greater or equal to t_o ? If the function F did not depend on x_{t-r} nor $\dot{x}_{t-r}$ the answer of course is that we have to know x_{t_o} only. Instead if the function F depends on x_{t-r} or/and $\dot{x}_{t-r}$ the answer is that we have to know the function x_t in the interval $[t_o - r, t_o]$. The configuration space associated with eq. (1.1) is therefore the infinite dimensional space of all sufficiently smooth function $x(t)$ defined in the preceding interval.

The general procedure to solve eq. (1.1) is the following: First of all, as we have just mentioned, we have to choose an arbitrary function in the interval $[t_o - r, t_o]$. Let us call this function $\varphi(t)$. This choice gives then a meaning to the function F in the interval $[t_o, t_o + r]$ on which it becomes a function of t and x_t. Therefore the equation (1.1) can be integrated in this latter interval as an ordinary differential equation. The general solution $x(t;C)$ will depend on an arbitrary constant C which will be determined by the condition:

$$\varphi(t_o) = x(t_o, C) \tag{1.2}$$

the corresponding solution gives then a meaning to the function F in the interval $[t_o + r, t_o + 2r]$ and the procedure can be iterated. This method to integrate eq- (1.1) is called the method of steps.

Clearly enough the solutions $x(t)$ thus obtained will have a discontinuous derivative $\dot{x}(t)$ at the time t_o because there is no

relationship, but accidental, between the past derivative at t_o which depends on the initial arbitrary function $\varphi(t)$ only and the future derivative which depends on $\varphi(t)$ and the function F . If eq. (1.1) is of the neutral type, i.e., if F depends on $\dot{x}_{t-r}$ the derivative will be discontinuous also at the time $t_o + nr$ for all integers n . On the contrary if eq. (1.1) is of the pure retarded type, i.e., if the function F does not depend on $\dot{x}_{t-r}$:

$$\dot{x}_t = F(t, x_t; x_{t-r}) \tag{1.3}$$

then the situation is as follows: there will be in general a discontinuity of $\dot{x}(t)$ at the time t_o but this derivative will be continuous from there onwards. More generally: the i-th derivative $x_t^{(i)}$ will be discontinuous at the times t_o , $t_o + r$,...., $t_o + (i-1)r$ and continuous from this latter time onwards. This assertion follows very simply from eq. (1.3) and from the equations that we obtain from it by calculating the successive derivatives with respect to t :

$$x_t^{(i)} = F^{(i-1)}(t, x_t; x_{t-r}, \dot{x}_{t-r}, \ldots, x_{t-r}^{(i-1)}) \tag{1.4}$$

this regularization in the future of the solutions obtained by the method of steps is an important property of the pure retarded equations. We shall see below that, at least in some cases, this property is the first step towards a deeper regularization.

Important examples of pure retarded differential equations are the linear ones:

$$\dot{x}_t = G\, x_t + H\, x_{t-r} \tag{1.5}$$

G and H being constants. These equations have analytic elementary solutions in the interval $]-\infty\ ,\ +\infty[$ of the form:

$$x_t = e^{\alpha t} \tag{1.6}$$

α being a solution of the characteristic equation, i.e., the equation which one obtains substituting (1.6) into (1.5) :

$$\alpha = G + H\, e^{-\alpha r} \tag{1.7}$$

The main results concerning these equations are the following (we invite the reader to consult Bellman and Cooke's book for a complete and rigorous presentation of the subject): i) the characteristic equation has an infinite number of complex roots; ii) among the complex roots there exists always one which has the largest real part (this

would be false if the equation had been of the neutral type); iii) the general solution of eq. (1.5) obtained by the method of steps starting from an arbitrary initial function $\varphi(t)$ defined in the interval $[-r, o]$ can be represented as a linear combination of an infinite number of elementary solutions with coefficients which in general will be polynomials having a degree equal to the order of multiplicity of the corresponding characteristic root.

2. The Concept of Attractor

Let us consider again a hereditary equation of the type (1.1) and let us consider an ordinary differential equation of order J :

$$x_t^{(J)} = \mathcal{R}(t, x_t, \dot{x}_t, \dots, x_t^{(J-1)}) \tag{2.1}$$

We shall say that eq. (2.1) is a Reduction of order J of the hereditary eq. (1.1) if all the solutions of eq. (2.1) are solutions of equation (1.1). In some sense hereditary equations can be considered as ordinary equations of infinite order and therefore we might expect them to have reductions of any order.

Definition. We shall say that a Reduction of a hereditary equation is an Attractor if its solutions can be imagined, in an "appropriate sense", as the asymptotes of the solutions of the hereditary equation for generic initial conditions.

This definition is meant to give some general but only intuitive meaning to the concept of Attractor. But actually the "appropriate sense" we have referred to has to be made precise for each particular case.

Let us consider the following linear equation:

$$\dot{x}_t = H x_{t-1} \tag{2.2}$$

The characteristic equation is:

$$\alpha = H e^{-\alpha} \tag{2.3}$$

which has a simple real root if $H > -1/e$. Let us assume that this is the case. Then:

$$x_t = A e^{\alpha t} \tag{2.4}$$

where A is an arbitrary constant, is a solution of eq. (2.2) in the interval $]-\infty, +\infty[$. On the other hand expression (2.4) is the general solution of the following first order differential equation:

$$\dot{x}_t = \alpha x_t \tag{2.5}$$

Therefore, when α defined by the characteristic equation (2.3) exists, eq. (2.5) is a reduction of order 1 of eq. (2.2).

We have integrated numerically by the method of steps eq. (2.2), considering a variety of initial conditions $\varphi(t)$ in the interval $]-1\ ,\ 0[$ including linear functions, sinuisoidal, exponential or even aleatory data. The behaviour which has systematically been observed, for values of H greater than $-1/e$ and not too large is that beyond a certain time the quantity $\beta_t = \dot{x}_t/x_t$ defined by the numerical solution tends very rapiddly, as t increases, to a constant which is, up to the precision of the numerical integration, the real root α of the characteristic equation (2.3). It is this result which we interpret as saying that eq. (2.5) is an Attractor of eq. (2.2).

Let us consider the following second example:

$$\dot{x}_t = G\,(x_t - x_{t-r}) \tag{2.6}$$

The characteristic equation is:

$$\alpha = G(1 - e^{-\alpha}) \tag{2.7}$$

This equation admits for any value of G the solution $\alpha = 0$ and if G is positive it admits a second real root α which is negative if $G < 1$ and it is positive if $G > 1$. Therefore whatever the value of G :

$$x_t = A \qquad \text{A: const.} \tag{2.8}$$

is a solution of eq. (2.6) and if $G > 0$ then:

$$x_t = A + B\,e^{\alpha t} \qquad \text{A,B : const.} \tag{2.9}$$

is also a real solution of eq. (2.6). From eq. (2.8) it follows that:

$$\dot{x}_t = 0 \tag{2.10}$$

is a Reduction of order 1 of eq. (2.6) for all values of G and eliminating the constants A and B from eq. (2.9) and its first and second derivatives it follows that for $G > 0$:

$$\ddot{x}_t = \alpha \dot{x}_t \tag{2.11}$$

is a Reduction of order 2 of eq. (2.6).

As for eq. (2.2) it would be of course very easy to construct Re-

ductions of any order of eq. (2.6). The two Reductions (2.10) and (2.11) have the particular supplementary interest that they are Attractors. In fact a numerical integration under the same general conditions as in the preceeding example has shown that for $G<1$ the quantity $\dot{x}_t$ calculated from the numerical solution, beyond a certain time tends to zero as t increases. We interpret this numerical result as meaning that eq. (2.10) is an Attractor of eq. (2.6). On the other hand if $G>0$ it can be seen that the quantity $\beta_t = \ddot{x}_t/\dot{x}_t$ beyond a certain time tends as t increases towards a constant α which is, up to the precision of the numerical integration equal to the non-zero real solution of the characteristic equation (2.7). We say then that eq. (2.11) is an Attractor of eq. (2.6) when $G>0$. Notice that in the interval $G \in]0,1[$ both eqs. (2.10) and (2.11) can be considered simultaneously as Attractors of eq. (2.6). This can be so because for $\alpha < 0$ both function (2.8) and (2.9) have the same asymptotic behaviour.

Of course for these two linear examples the concept of Attractor could be made precise, as it could be proved exactly that eqs. (2.5) and eqs. (2.10) and (2.11) are respectively Attractors of eqs. (2.2) and (2.6). The properties which we have mentioned at the end of the preceding paragraph give the hint for the definition and the proofs. But here we have preferred to remain at the level of intuition and conviction which can be attained by numerical calculations because this is the level at which we can raise ourselves in the discussion of highly more complicated problems. It is our feeling that the time is not ripe for a rigorous analysis.

The importance for a hereditary equation to have a known Attractor is obvious. Not only does this means that we can construct some of its smoothest solutions it can have in the largest possible time intervals; it means also that, if we are not interested in the near future of the solutions after the end of the initial constrained motion but we are interested only on their behaviour after a while, the Attractor can be considered as a useful substitute of the hereditary equation.

3. The Associated Predictive Differential Equation

Let us assume that the function F of eq. (1.3) is proportional to a coupling constant G :

$$\dot{x}_t = G \cdot W(t, x_t; x_{t-r}) \tag{3.1}$$

For this class of equations, which could be extended of course, we

shall introduce the concept of Associated Predictive Differential Equation (P.D.E.) according to the following definition.

Definition. Let us consider an ordinary first order differential equation depending on the coupling constant G

$$\dot{x}_t = \mathfrak{Z}(t, x_t ; G) \tag{3.2}$$

We shall say that eq. (3.2) is a P.D.E. associated with the hereditary equation (3.1) if i) eq. (3.2) is a Reduction of eq. (3.1) and ii) for small values of G the function $\mathfrak{Z}$ can be developped in power series of G of the following form:

$$\mathfrak{Z}(t, x_t ; G) = G \cdot \overset{(1)}{\mathfrak{Z}}(t, x_t) + G^2 \cdot \overset{(2)}{\mathfrak{Z}}(t, x_t) + \dots, \tag{3.3}$$

i.e., without a zero power term.

According to this definition eq. (2.5) is a P.D.E. associated with eq. (2.2) and eq. (2.10) is a P.D.E. associated with eq. (2.6).

For equations more complicated than eqs. (2.2) or (2.6) it might be difficult or impossible to obtain corresponding associated P.D.E. Therefore it is important to set forth a perturbation algorithm to construct approximate ones. To do it we shall make some supplementary assumptions. To each hereditary equation satisfying them, it will correspond then to one and only one formal associated P.D.E. Let us write:

$$\dot{x}_t^{[k]} = G\, \mathfrak{Z}^{(1)} + G^2 \mathfrak{Z}^{(2)} + \dots + G^K \mathfrak{Z}^{(K)} \tag{3.4}$$

For each value of k each of these approximate P.D.E. can be integrated backwards considering x_t as initial condition corresponding to time t . Let $x_{t-r}^{\{k\}}(t, x_t ; G)$ be the function which gives the value of the solution of (3.4) at time $t-r$ as a function of t, x_t and G . We assume that these functions can be developped also as a power series of G in the following form:

$$x_{t-r}^{\{k\}} = x_t + G \cdot \rho^{\{k,1\}}(t, x_t) + G^2 \cdot \rho^{\{k,2\}} + \dots \tag{3.5}$$

and that

$$\rho^{\{k,i\}} = \rho^{\{k',i\}} \qquad \text{if} \qquad i \le \inf(k, k') \tag{3.6}$$

in which case we shall write

$$\rho^{\{k,i\}} \equiv \rho^{(i)} \tag{3.7}$$

Substituting then the formal expansion:

$$x_{t-r}^{[s]} = x_t + G\rho^{(1)} + G^2\rho^{(2)} + \ldots + G^s \cdot \rho^{(s)} \qquad (3.8)$$

into eq. (3.1), expanding and identifying with eq. (3.3) gives the necessary relations to identify the coefficients $\mathfrak{Z}^{(n)}$. The first one is:

$$\mathfrak{Z}^{(1)}(t, x_t) = W(t, x_t; x_t) \qquad (3.9)$$

This is the only trivial step of this algorithm. The next one would consist in calculating $\rho^{(1)}$ by integrating the first order approximate P.D.E. (3.4). The feasibility of working this out exactly will depend of course on each particular case. But $\rho^{(1)}$ as $\rho^{(n)}$ can always be calculated at the appropriate level of approximation.

This method applied to eq. (2.6) leads trivially to eq. (2.10). Applied to eq. (2.2) it leads to an infinite series whose sum is eq. (2.5). Eqs. (2.5) and (2.10) are therefore the P.D.E.'s associated respectively with eqs. (2.2) and (2.6).

Let us consider the following hereditary equation:

$$\dot{x}_t = 2G \Big/ (x_t + x_{t-r}) \qquad (3.10)$$

According to eq. (3.9) the first order P.D.E. associated to it is

$$\dot{x}_t^{[1]} = G/x_t \qquad (3.11)$$

Integrating this equation we obtain for sufficiently large positive values of x_t :

$$x_{t-r}^{[1]} = (x_t^2 - 2Gr)^{1/2} \qquad (3.12)$$

Assuming now that:

$$\frac{Gr}{x_t^2} < 1 \qquad (3.13)$$

we obtain

$$x_{t-r}^{[1]} = x_t \left(1 - \frac{Gr}{x_t^2}\right) \qquad (3.14)$$

and the second order associated P.D.E.

$$\dot{x}_t^{[2]} = \frac{G}{x_t}\left(1 + \frac{Gr}{2x_t^2}\right) \qquad (3.15)$$

We expect of course this equation to be a good approximation of an exact P.D.E., supposed to exist, for large values of x_t only.

We have integrated numerically eq. (3.10) for a variety of initial conditions and we have examined more particularly the evolution of

the quantity:

$$\beta_t = \frac{\dot{x}_t - \dot{x}_t^{[2]}}{\dot{x}_t} \tag{3.16}$$

We have seen that when the quantity (3.13) remains small compared to 1 that beyond a certain time the quantity β_t tends to zero. We interpret this result as saying that in the appropriate domain of configuration space the approximate equation (3.15) is an Attractor of eq. (3.10).

We have considered three examples of first order hereditary differential equations, namely eq. (2.2), eq. (2.6) and eq. (3.10) of the type of eq. (3.1). In the three cases we have seen that the corresponding associated P.D.E. (second order approximation for eq. (3.10)) was an Attractor in an appropriate, self explanatory sense. We consider this finding, which has its root in the very meaning of the perturbation constructive method we have presented, as an indication that one of the most relevant questions that we can ask in connection with a hereditary equation is the following: Is it possible to construct the associate P.D.E. (exactly or approximate)? If yes, is the associate P.D.E. an Attractor?

SECTION II : THE ELECTROMAGNETIC INTERACTION

1. Equations of motion of two charges

Let us consider two point-like electric charges e_a (a=1,2) with masses m_a and let:

$$L_a : \qquad x_a^\alpha = x_a^\alpha(\tau_a) \qquad (\alpha,\beta,\ldots = 0,1,2,3) \tag{1.1}$$

τ_a being the proper times, be the parametric equations of their time-like future oriented world lines. We shall use the signature +2 of Minkowski space-time and a system of units such that $c = 1$. Therefore we shall have:

$$u_a^\alpha u_{a\alpha} = -1 \quad , \qquad u_a^\alpha \equiv \frac{dx_a^\alpha}{d\tau_a} \tag{1.2}$$

and $u_a^{\,0} > 0$ by definition of future oriented.

If no other interaction nor constraint, besides the electromagnetic interaction, is acting upon the charges; if we assume causality; and if we neglect radiation reaction forces (later on we shall take them into account), then the functions (1.2) must be for τ_a large enough, i.e. beyond the initial constrained motion, solutions of the following system of hereditary second order differential equations

$$\frac{du_a^\alpha}{d\tau_a} \equiv \xi_a^\alpha = W_a^\alpha \tag{1.3}$$

where:

$$W_a^\alpha \equiv e_a e_{a'} m_a^{-1} \hat{r}_{aa'}^{-2} \Big\{ \hat{r}_{aa'}^{-1} \Big[1 + (\hat{l}_{aa'} \hat{\xi}_{aa'}) \Big] \big(\hat{k}_{aa'} \hat{l}_{aa'}^\alpha - \hat{\lambda}_{aa'} \hat{u}_{aa'}^\alpha \big) - \\ - (\hat{\xi}_{aa'} u_a) \hat{l}_{aa'}^\alpha - \hat{\lambda}_{aa'} \hat{\xi}_{aa'}^\alpha \Big\} \tag{1.4}$$

where:

$$a' \neq a \quad ; \qquad \hat{l}_{aa'}^\alpha \equiv \hat{x}_a^\alpha - \hat{x}_{aa'}^\alpha$$

$$\hat{r}_{aa'} \equiv -(\hat{l}_{aa'} \hat{u}_{aa'}) \quad , \qquad \hat{\lambda}_{aa'} \equiv -(\hat{l}_{aa'} u_a) \tag{1.5}$$

$\hat{x}_{aa'}^\alpha$ being the intersection of the world line of particle a' with the past light cone with vertex at the point x_a^α, $\hat{u}_{aa'}^\alpha$ and $\hat{\xi}_{aa'}^\alpha$ being the unit four-velocity and four-acceleration at the point $\hat{x}_{aa'}^\alpha$, and round brackets indicating scalar products.

Equation (1.3) express that each particle a obeys the Lorentz equations of motion corresponding to the retarded field created by particle a'. They are hereditary equations as are the ones we have considered in the first section but highly more complicated: they involve a larger number of variables; they are of second order; they are a system; they are of neutral type (the functions W_a^α contain the $\hat{\xi}_{aa'}^\alpha$); and the retardation is of a functional type. Nevertheless

as we shall see we can use similar concepts and apply similar methods to analyse these equations.

The method of steps can be applied to construct solutions of class C^1 of equations (1.3). For this it is necessary to enlarge the system of equations (1.3). Let $\hat{\tau}_{aa'}$ be the value of $\tau_{a'}$ corresponding to the point $\hat{x}^\alpha_{aa'}$. This value of $\hat{\tau}_{aa'}$ is a function of τ_a and deriving the relation:

$$[x^\alpha_a(\tau_a) - x^\alpha_{a'}(\hat{\tau}_{aa'})]\cdot[x_{a\alpha}(\tau_a) - x_{a'\alpha}(\hat{\tau}_{aa'})] = 0 \tag{1.6}$$

with respect to τ_a we obtain that this function has to satisfy the following differential equation:

$$\frac{d\hat{\tau}_{aa'}}{d\tau_a} = \hat{\lambda}_{aa'}\hat{r}^{-1}_{aa'} \tag{1.7}$$

which we consider simultaneously with eqs. (1.3). Let us assume now that we are given two pieces of time-like future oriented world lines with parametric equations:

$$\Gamma_a: \qquad x^\alpha_a = \varphi^\alpha_a(\tau_a) \tag{1.8}$$

in the intervals $\tau_a \in [\tau_a^+, \tau_a^-]$, $\tau_a^\pm$ being such that:

$$[\varphi^\alpha_a(\tau_a^+) - \varphi^\alpha_{a'}(\tau_{a'}^-)]\cdot[\varphi_{a\alpha}(\tau_a^+) - \varphi_{a'\alpha}(\tau_{a'}^-)] = 0 \tag{1.9}$$

Γ_b give a meaning to the right-hand sides of eqs. (1.3) and (1.7) as functions of x^α_a, u^α_a and $\hat{\tau}_{aa'}$, as long as the past light cone with vertex at the point x^α_a intersects $\Gamma_{a'}$. Therefore in the corresponding τ_b intervals these equations can be integrated as a system of ordinary differential equations. The initial conditions for x^α_a are of course $\varphi^\alpha_a(\tau_a^+)$; those for u^α_a are taken to be $\left(d\varphi^\alpha_a/d\tau_a\right)(\tau_a^+)$ to have C^1 solutions; and the initial condition for $\hat{\tau}_{aa'}$ is necessarily $\tau_{a'}^-$. We obtain thus two new pieces of world lines which can be used as initial data for a new step.

The equations of motion of two charges are in fact simpler in a particular well-known case. Let us assume that the initial world lines Γ_a lie both on a single time-like 2-plane, i.e., the initial space motion is restricted to one rectilinear space dimension. Obviously then the entire solution (both Γ_a) will lie on the same 2-plane. This means in particular that $\hat{\xi}^\alpha_{aa'}$ will be a linear combination of $\hat{l}^\alpha_{aa'}$ and $\hat{u}^\alpha_{aa'}$:

$$\hat{\xi}^\alpha_{aa'} = \lambda\,\hat{l}^\alpha_{aa'} + \mu\,\hat{u}^\alpha_{aa'} \tag{1.10}$$

The substitution of such expressions into eqs. (1.4) yields the following result:

$$W_a^{\alpha} = e_a e_{a'} m_a^{-1} \hat{r}_{aa'}^{-3} \left(\hat{k}_{aa'} l_{aa'}^{\alpha} - \hat{s}_{aa'} \hat{\mu}_{aa'}^{\alpha}\right) \qquad (1.11)$$

therefore for rectilinear motion the hereditary electromagnetic dynamical system (1.3) is much simpler than in the general case. The simplification comes in by the fact that the number of variables is reduced but also by the fact that the expressions (1.11) no longer depend on the retarded accelerations and therefore the corresponding dynamical system is of the pure retarded type.

Because of this result an analysis similar to the one we made in the preceding section proves that in this particular case there is a regularization in the future of the solutions obtained by the method of steps in the sense that after the i-th step the solution is of class C^{i+1}.

2. Regularization of the solutions in the future

In the general case where the dynamical system (1.3) is of the neutral type there is of course no exact regularization. Nevertheless we shall see in this paragraph that even in this case there exists an approximate regularization in at least two meanings.

Let us use the notation $\overline{\overline{\beta}}_a^{\alpha}$ to indicate the discontinuity of β_a^{α} at the end point on L_a of one step, and let us use the notation $\hat{\overline{\beta}}_{aa'}$ to indicate the discontinuity of the acceleration at the corresponding retarded point. From eqs. (1.3) and (1.4) it follows that:

$$\overline{\overline{\beta}}_a^{\alpha} = -\hat{K}_{aa'} \left\{ \hat{\overline{\beta}}_{aa'}^{\alpha} + \hat{r}_{aa'}^{-1} \left(\hat{l}_{aa'} \hat{\overline{\beta}}_{aa'}\right) \hat{\mu}_{aa'}^{\alpha} - \hat{s}_{aa'}^{-1} \left[\hat{r}_{aa'}^{-1} k_{aa'} \left(\hat{l}_{aa'} \hat{\overline{\beta}}_{aa'}\right) - \left(u_a \hat{\overline{\beta}}_{aa'}\right)\right] \hat{l}_{aa'}^{\alpha} \right. \qquad (2.1)$$

where:

$$\hat{K}_{aa'} = e_a e_{a'} m_a^{-1} \hat{r}_{aa'}^{-2} \hat{s}_{aa'} \qquad (2.2)$$

Considering the square of both members of eqs. (2.1) and taking into account that:

$$\left(\hat{\overline{\beta}}_{aa'} \hat{\mu}_{aa'}\right) = 0 \qquad \hat{l}_{aa'}^{2} = 0 \qquad (2.3)$$

we obtain:

$$\overline{\overline{\beta}}_a^{2} = \hat{K}_{aa'}^{2} \left[\hat{\overline{\beta}}_{aa'}^{2} - \hat{r}_{aa'}^{-2} \left(\hat{l}_{aa'} \hat{\overline{\beta}}_{aa'}\right)^2\right] \qquad (2.4)$$

from which it follows the inequality:

$$\left|\overline{\overline{\beta}}_a\right| \leq \hat{K}_{aa'} \left|\hat{\overline{\beta}}_{aa'}\right| \qquad (2.5)$$

$\hat{\overline{3}}^{\alpha}_{aa'}$ is the discontinuity of $\overline{3}^{\alpha}_{a'}$ just one step below and therefore the preceding formula can be iterated. Using the notation $\overline{3}^{\alpha}_{a_i}$ to indicate the discontinuity of $\overline{3}^{\alpha}_{a}$ at the end of the i-th step and similar self-explanatory notations $\hat{K}_{a_n a'_{n-1}}$ we can write:

$$|\overline{3}_{a_n}| \leq \hat{K}_{a_n a'_{n-1}} \cdots \hat{K}_{a'_1 a_0} |\overline{3}_{a_0}| \tag{2.6}$$

if n is even, and:

$$|\overline{3}_{a_n}| \leq \hat{K}_{a_n a'_{n-1}} \cdots \hat{K}_{a_1 a'_0} |\overline{3}_{a'_0}| \tag{2.7}$$

if n is odd. The quantities $K_{aa'}$ are the ratio of two quantities: $e_a e_{a'} m_a^{-1}$ and $\hat{r}^2_{aa'} \cdot \hat{\lambda}^{-1}_{aa'}$. The first ones are natural lengths associated with each charge. The second ones are sort of distances between the particles (they would be exactly the proper distance between the two charges if these were constrained to be at relative rest). It turns out that the $\hat{K}_{aa'}$ are very small compared to 1 as long as the distance between the particles does not become extremely small. Therefore eq. (2.7) tells us that the quantities $|\overline{3}_{a_n}|$ will tend numerically very rapidly towards zero. Since $\overline{3}^{\alpha}_{a}$ and therefore $\overline{3}^{\alpha}_{a}$ are space-like vectors we can conclude that these discontinuities themselves will tend to zero, even if strictly speaking they will never be zero in the generic case.

For n large enough we can assume that the accelerations are continuous. Assuming this and deriving eqs. (1.4) with respect to τ_a we can use a similar argument to discuss the evolution of the discontinuities of the derivatives of the accelerations. This leads to the conclusion that these discontinuities tend to zero also as n increases; a conclusion which can then be extended obviously to the discontinuities of the derivatives of any order. We have therefore a regularization of the solutions in the future.

There is another approach which leads to the conclusion that there is an approximate regularization in the future of the solutions of eqs. (1.3). Let us assume that we have integrated these equations by the method of steps and let us consider expressions (1.4) beyond the first step. The accelerations $\hat{\overline{3}}^{\alpha}_{aa'}$ will then be a function of the coupling constant $e_a e_{a'}$, vanishing when this constant vanishes. We shall assume that $\hat{\overline{3}}^{\alpha}_{aa'}$ can be represented by a power series of $e_a e_{a'}$ of the following type:

$$\hat{\overline{3}}^{\alpha}_{aa'} = \hat{\overline{3}}^{\alpha\,(1,1)}_{aa'} + \hat{\overline{3}}^{\alpha\,(2,2)}_{aa'} + \ldots + \hat{\overline{3}}^{\alpha\,(n,n)}_{aa'} + \ldots \tag{2.8}$$

the n-th term being proportional to $e_a^n e_{a'}^n$. Substituting this power

series into expressions (1.4) and keeping first order terms only we obtain:

$$W_a^{\alpha(1,1)} = e_a e_{a'} \cdot m_a^{-1} \hat{r}_{aa'}^{-3} \left(\hat{k}_{aa'} \hat{l}_{aa'}^{\alpha} - \hat{s}_{aa'} \cdot \hat{u}_{aa'}^{\alpha} \right) \qquad (2.9)$$

These equations are identical with eqs. (1.11) except that here the variables are not restricted to one dimensional rectilinear motion. The dynamical system (1.3) corresponding to the approximation (2.9) is of the pure retarded type and therefore we know that there will be a regularization in the future of its solutions.

This conclusion can be extended to a higher approximation of system (1.3). In fact, let us consider the second order approximation of expressions (1.4), i.e., let us keep the first order terms $\hat{\mathfrak{Z}}_{aa'}^{\alpha}(1,1)$ on them. From eqs. (2.9) we know that <u>beyond the second step</u> these quantities can be written as :

$$\hat{\mathfrak{Z}}_{aa'}^{\alpha(1,1)} = e_a e_{a'} m_{a'}^{-1} \cdot \hat{r}_{aa'a}^{-3} \left(\hat{k}_{aa'a} \hat{l}_{aa'a}^{\alpha} - \hat{s}_{aa'a} \hat{u}_{aa'a}^{\alpha} \right) \qquad (2.10)$$

where:

$$\hat{l}_{aa'a}^{\alpha} = \hat{x}_{a a'}^{\alpha} - \hat{x}_{aa'a}^{\alpha} \quad ; \quad \hat{k}_{aa'a} = -(\hat{u}_{aa'} \hat{u}_{aaa'})$$
$$\hat{r}_{aa'a} = -(\hat{l}_{aa'a} \hat{u}_{aa'a}) \quad ; \quad \hat{s}_{aa'a} = -(\hat{l}_{aa'a} \hat{u}_{aa'}) \qquad (2.11)$$

$\hat{x}_{aa'a}^{\alpha}$ being the intersection of the world-line L_a with the past null cone with vertex at the point $\hat{x}_{aa'}^{\alpha}$, and $\hat{u}_{aa'a}^{\alpha}$ being the unit four-velocity of particle a at the point $\hat{x}_{aa'a}^{\alpha}$. Let us consider the straight-line passing through the point x_a^{α} in the direction of u_a^{α} and let $\hat{x}_{aa'a}^{*\alpha}$ be the intersection of this straight-line with the past null cone having its vertex at the point $\hat{x}_{aa'}^{\alpha}$. Since when $e_a e_{a'}$ vanishes the two points $\hat{x}_{aa'a}^{\alpha}$ and $\hat{x}_{aa'a}^{*\alpha}$ coincide we can assume that these components differ by quantities which are of first order, and thus we can use $\hat{x}_{aa'a}^{*\alpha}$ instead that $\hat{x}_{aa'a}^{\alpha}$ in eqs. (2.10). A similar argument proves also that we can use u_a^{α} instead of $\hat{u}_{aa'a}^{\alpha}$. These two substitutions lead to the following modification of eqs. (2.10)

$$\hat{\mathfrak{Z}}_{aa'}^{\alpha(1,1)} = e_a e_{a'} m_{a'}^{-1} \cdot \hat{s}_{aa'}^{-3} \left(-\hat{k}_{aa'} \hat{l}_{aa'}^{\alpha} + \hat{r}_{aa'} u_a^{\alpha} \right) \qquad (2.12)$$

and we obtain finally the following second order approximation for expressions (1.4):

$$W_a^{\alpha[2]} = W_a^{\alpha(1,1)} + e_a^2 e_{a'}^2 m_a^{-1} m_{a'}^{-1} \hat{r}_{aa'}^{-1} \hat{s}_{aa'}^{-3} \left(\hat{l}_{aa'}^{\alpha} - \hat{s}_{aa'} u_a^{\alpha} \right)$$
$$- e_a e_{a'} m_a^{-1} \hat{s}_{aa'}^{-2} \hat{r}_{aa'} W_a^{\alpha(1,1)} \qquad (2.13)$$

where the notation in the left-hand-side means that all terms up to

second order have been included.

The system of equations (1.3) using expressions (2.13) as an approximation of expressions (1.4) is again of the non neutral once retarded type. Therefore we know that at this approximation the solutions obtained by the method of steps will become smoother and smoother in the future.

3. Radiation reaction forces

The concept of Order Reduction is partially contained already in Landau's and Lifshitz book [4] in connection with the Lorentz-Dirac equation of a charge in an external field, but as far we know the first paper where the concept was clearly introduced in connection with the preceding problem but also in connection with the two-body electromagnetic problem, was in Kerner's paper [5]. We are going to present this concept in a simplified context using the point of view of Sanz [6], [7].

Let us consider the following ordinary differential equation:

$$\dot{x}_t = G\left[W(t, x_t) + \ddot{x}_t\right] \tag{3.1}$$

where G is some coupling constant and W a sufficiently smooth function of its arguments. Eq. (3.1) is a second order differential equation. But equivalently we can say that eq. (3.1) represents a family of <u>first</u> order equations

$$\dot{x}_t = \zeta(t, x_t; G) \tag{3.2}$$

ζ being the general solution of the partial differential equation:

$$\zeta = G\cdot\left(W + \frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x}\cdot\zeta\right) \tag{3.3}$$

Let us assume that a physical quantity x has to evolve in accordance with eq. (3.1). If we are sure that no other condition restricts the variable x then our job would be, say, to find the solutions of eq. (3.1) directly, or to find first the solutions of eq. (3.3) and to find then the solutions of eqs. (3.2). On the contrary if we have other physical constraints on the system then eq. (3.3) will have to be considered as a condition <u>among others</u> to determinate the eq. (3.2) which will describe our physical system. Let us assume for example that $\dot{x}_t$ is a measurable quantity which by its very meaning or because of other principles of the theory has to vanish with G or, say, has to be a very smooth function of G. If we express this condition by saying that the function ζ can be represented by a power series of G :

$$\zeta = G\cdot\zeta^{(1)} + G^2\zeta^{(2)} + \dots \tag{3.4}$$

then substituting this power series into eq. (3.3) gives all the coefficients without any ambiguity. The first ones being for instance:

$$\left.\begin{aligned} &\zeta^{(1)} = W \qquad \zeta^{(2)} = \partial W/\partial t \\ &\zeta^{(3)} = W\cdot\frac{\partial W}{\partial x} + \frac{\partial^2 W}{\partial t^2} \end{aligned}\right\} \tag{3.5}$$

Of course for some simple cases an exact construction is possible. Let us consider, for instance, the differential equation:

$$\dot{x}_t = G\cdot(2t + \ddot{x}_t) \tag{3.6}$$

The complete family of first order differential equations equivalent to the second order one is:

$$\dot{x}_t = 2G(t+G) + C\cdot e^{t/G} \tag{3.7}$$

where C is an arbitrary constant. Among these equations the only one having the property of having a second member analytic in G in the neighbourhood of $G = 0$ is :

$$\dot{x}_t = 2G(t+G) \tag{3.8}$$

The process which consists in substituting eq. (3.1) by eq. (3.2) with the right-hand-side being a solution of eq. (3.3) and being analytic in G in the neighbourhood of $G = 0$ is called the Order Reduction of the equation: of course this concept can be used in more general cases including the hereditary equations (which as we mentioned in the first section, can be formally considered as ordinary equations of infinite order) and, as we shall see in a moment, the dynamical system of two electric charges when the radiation reaction forces are taken into account.

We know that when the back reaction of the radiation on two electric charges cannot be neglected the dynamical system (1.3) has to be replaced by:

$$\frac{du_a^\alpha}{d\tau_a} = W_a^\alpha + \frac{2}{3}m_a^{-1}e_a^2\left(\delta_\beta^\alpha + u_a^\alpha\cdot u_{a\beta}\right)\frac{d^2u_a^\beta}{d\tau_a} \tag{3.9}$$

with W_a^α given by expressions (1.4). This system of equations is of third order but eqs. (3.9) are by no means the only equations which govern the evolution of two charges. We believe that it would be rash to forget at this level of the very foundations on which relativistic dynamics is based. If one of the charges, say e_a , is non zero but

the other one $e_{a'}$ is zero then the two particles are uncoupled and they should, according to the Principle of Inertia move freely. This is not necessarily the case for particle a according to eqs. (3.9) and therefore the Principle of Inertia has to be enforced by a supplementary condition. The best way to do it is to say that the real dynamical system which governs the evolution of two charges is not the system (3.9) but its Order Reduced one. This is the point of view we take up here. It also takes into account this fundamental idea that second derivatives are related to forces, i.e., quantities which we expect to have measures depending smoothly on the intensity of the interaction.

Actually we shall not reduce the exact system (3.9) but

$$\frac{du_a^\alpha}{d\tau_a} = W_a^{\alpha[2]} + \frac{e}{3} m_a^{-1} e_a^2 \left(\delta_\beta^\alpha + u_a^\alpha u_{a\beta}\right) \frac{d^2 u_a^\beta}{d\tau_a^2} \equiv \tilde{W}_a^\alpha \tag{3.10}$$

with $W_a^{\alpha[2]}$ being the second approximation (2.13) of W_a^α which we have calculated in the preceding paragraph. Moreover we shall push the perturbative construction of the reduced system only up to the maximum order which is consistent with the approximation $W_a^{\alpha[2]}$. That is to say we shall calculate the terms which are at most proportional to the product of four charges. We could proceed as indicated at the beginning of this paragraph but we can also argue directly as follows. By its very structure the lowest order term of the right-hand-side of eqs. (3.10) is a term proportional to $e_a e_{a'}$, and therefore the lowest order term of the second term of the right-hand-side is proportional to $e_a^3 e_{a'}$. Therefore we can write directly that

$$\tilde{W}_a^{\alpha[2]} = W_a^{\alpha[2]} + \frac{e}{3} m_a^{-1} e_a^2 \left(\delta_\beta^\alpha + u_a^\alpha u_{a\beta}\right) \left(\frac{d}{d\tau_a} W_a^{\beta(1,1)}\right)^{(1,1)} \tag{3.11}$$

where $W_a^{\alpha(1,1)}$ is given by eqs. (2.9). The derivative with respect to τ_a is a total derivative, i.e., the variables which refer to particle a' are considered as functions of τ_a also through their dependence on $\hat{\tau}_{aa'}$. Using eqs. (1.7) a straightforward calculation gives:

$$\tilde{W}_a^{\alpha[2]} = W_a^{\alpha[2]} - e\, m_a^{-1} e_a^2 \hat{r}_{aa'}^{-1} \left(\hat{k}_{aa'} - \hat{r}_{aa'}^{-1} \hat{\lambda}_{aa'}\right) W_a^{\alpha(1,1)} + \\ + \frac{e}{3} m_a^{-2} e_a^3 e_{a'} \hat{r}_{aa'}^{-3} \left(\hat{k}_{aa'} \mu_a^\alpha - \hat{\mu}_{aa'}^\alpha\right) \tag{3.12}$$

In our opinion the system of differential equations:

$$\frac{du_a^\alpha}{d\tau_a} = \tilde{W}_a^{\alpha[2]} \tag{3.13}$$

is the dynamical system which has to be used to describe the electromagnetic interaction of two-point-like charges at the second order approximation. It includes the radiation reaction forces consistently at the same order. It is of the non neutral, once retarded, type and therefore we know that at this approximation its solutions obtained by the method of steps will become smoother amd smoother in the future.

4. Associated Predictive Poincaré Invariant System

The concept of a Predictive Poincaré Invariant System (P.I.S.) associated with a causal interaction, electromagnetic or other, has been available for some time. We refer the reader to [8], and the references therein, and to [9] and [10] for the case of the gravitational interaction. Nevertheless to make this paper self-contained we shall present briefly this concept again for the case of the electromagnetic interaction at the approximation that we have considered in the preceding paragraph.

Let us consider a two body newtonian-like dynamical system:

$$\dot{x}_a^i = v_a^i \quad , \quad \dot{v}_a^i = a_a^i (x_b^j, v_c^k; t) \qquad \left(\dot{-} \equiv \frac{d}{dt} -\right) \tag{4.1}$$

(a,b,c= 1,2 ; i,j,k,.. = 1,2,3) and let

$$x_a^i = \varphi_a^i (x_{bo}^j, v_{co}^k ; t) \tag{4.2}$$

with

$$\varphi_a^i (x_{bo}^j, v_{co}^k ; 0) = x_{ao}^i \quad ; \quad \dot{\varphi}_a^i (x_{bo}^j, v_{co}^k ; 0) = v_{ao}^i \tag{4.3}$$

be its general solution. Let us consider a galilean frame of reference of Minkowski space-time M_4 and let us consider the family of pairs of world lines depending on the 12 parameters (x_{ao}^i , v_{bo}^j) :

$$x_a^0 = t \quad ; \quad x_a^i = \varphi_a^i (x_{bo}^j, v_{co}^k ; t) \tag{4.4}$$

By definition the system of differential equations (4.1) is a P.I.S. if the family (4.4) is invariant under the usual realization of the Poincaré group acting on M_4 . It is known that a necessary [11], [12] and sufficient [13] condition for system (4.1) to be a P.I.S. is that the functions a_a^i satisfy the following system of non-linear partial differential equations (Currie-Hill equations):

$$\frac{\partial a_a^i}{\partial t} = 0 \quad ; \quad \epsilon^b \frac{\partial a_a^i}{\partial x_b^j} = 0$$

$$\delta^{s}_{\cdot\kappa j}\left(x^{\kappa}_{b}\cdot\frac{\partial a^{i}_{a}}{\partial x^{s}_{b}}+v^{\kappa}_{b}\cdot\frac{\partial a^{i}_{a}}{\partial v^{s}_{b}}\right)=\delta^{i}_{\cdot\kappa j}\,a^{\kappa}_{a}$$

$$v^{s}_{a}(x_{aj}-x_{bj})\cdot\frac{\partial a^{i}_{b}}{\partial x^{s}_{a}}+\left[v^{s}_{a}v_{aj}+a^{s}_{a}(x_{aj}-x_{bj})-\epsilon_{a}\delta^{s}_{j}\right]\frac{\partial a^{i}_{b}}{\partial v^{s}_{a}}=2v_{bj}a^{i}_{b}+v^{i}_{b}a_{bj} \tag{4.5}$$

where δ_{ijk} is the Levi-Civita tensor, $\epsilon_{b}=1$ and where we have applied Einstein's sommation convention for both types of indices.

Let us consider the following autonomous system of ordinary differential equations on M_4 .

$$\frac{dx^{\alpha}_{a}}{d\tau}=u^{\alpha}_{a}\quad,\qquad\frac{du^{\alpha}_{a}}{d\tau}=\Xi^{\alpha}_{a}(x^{\beta}_{b},u^{\gamma}_{c}) \tag{4.6}$$

We know [14] that a sufficient condition for this system to be locally equivalent to a P.I.S. is that the functions Ξ^{α}_{a} satisfy the Droz-Vincent equations [15] :

$$u^{\rho}_{a'}\cdot\frac{\partial\Xi^{\alpha}_{a}}{\partial x^{a'\rho}}+\Xi^{\rho}_{a'}\cdot\frac{\partial\Xi^{\alpha}_{a}}{\partial u^{a'\rho}}=0 \tag{4.7}$$

the constraints:

$$\Xi^{\rho}_{a}\cdot u_{a\rho}=0 \tag{4.8}$$

and the equations expressing that they are vector functions of vector arguments:

$$\epsilon_{b}\frac{\partial\Xi^{\alpha}_{a}}{\partial x^{\beta}_{b}}=0$$

$$\frac{\partial\Xi^{\alpha}_{a}}{\partial x^{\sigma}_{b}}x_{b\rho}-\frac{\partial\Xi^{\alpha}_{a}}{\partial x^{\rho}_{b}}\cdot x_{b\sigma}+\frac{\partial\Xi^{\alpha}_{a}}{\partial u^{\rho}_{b}}u_{b\sigma}-\frac{\partial\Xi^{\alpha}_{a}}{\partial u^{\sigma}_{b}}u_{b\rho}=\Xi_{a\sigma}\delta^{\alpha}_{\rho}-\Xi_{a\rho}\delta^{\alpha}_{\sigma} \tag{4.9}$$

The correspondence between the functions a^{i}_{a} and Ξ^{α}_{a} is as follows. If the latter are known we obtain the a^{i}_{a} by the formulae:

$$a^{i}_{a}(x^{j}_{b},v^{k}_{c})=(1-v^{2}_{a})\left(\bar{\Xi}^{i}_{a}-v^{i}_{a}\bar{\Xi}^{o}_{a}\right) \tag{4.10}$$

where the notation $\bar{\Xi}^{\alpha}_{a}$ means that the variables x^{α}_{a} and u^{β}_{b} have been restricted to the values:

$$x^{o}_{a}=t\quad,\qquad\bar{x}^{i}_{a}=x^{i}_{a}$$

$$\bar{u}^{o}_{a}=(1-v^{2}_{a})^{-1/2}\quad,\qquad\bar{u}^{i}_{a}=\bar{u}^{o}_{a}v^{i}_{a} \tag{4.11}$$

In a contrary direction if we know the a^{i}_{a} we obtain the Ξ^{α}_{a} by the formulae:

$$\Xi^{i}_{a}(x^{\beta}_{b},u^{\gamma}_{c})=(u^{o}_{a})^{2}(\delta^{i}_{r}+u^{i}_{a}u_{ar})\,\ddot{\varphi}^{r}_{a}(x^{j}_{bo},v^{k}_{co};x^{o}_{a})$$

$$\Xi^{o}_{a}(x^{\beta}_{b},u^{\gamma}_{c})=(u^{o}_{a})^{3}u_{ar}\,\ddot{\varphi}^{r}_{a}(x^{j}_{bo},v^{k}_{co};x^{o}_{a}) \tag{4.12}$$

the $(x^j_{bo}\ ,\ v^k_{co})$ being those functions of $(x^\beta_b\ ,\ u^\gamma_c)$ which we would obtain solving the following equations:

$$\left.\begin{aligned} x^i_a &= \varphi^i_a\,(x^j_{bo}\,;\,v^k_{co}\,;\,x^o_a) \\ (u^o_a)^{-1}u^i_a &= \dot{\varphi}^i_a\,(x^j_{bo}\,,v^k_{co}\,;\,x^o_a) \end{aligned}\right\} \qquad (4.13)$$

Any P.I.S. can thus be discussed using two formalisms. We call the original one for which the system is a Newtonian-like one the Manifestly Predictive Formalism. This is in our opinion the formalism of reference for any physical interpretation because it makes transparent that the space of initial conditions (the co-phase space) is twelve dimensional, i.e., the formalism does not use spurious degrees of freedom. We call the equivalent version of it for which the system is of the type (4.6) the Manifestly Invariant Formalism. The conditions (4.7) and (4.8) are not necessary to have equivalence between a Manifestly Predictive P.I.S. and a Poincaré Invariant System of equations of the type (4.6). There exists other versions of the Manifestly Invariant formalism (See for instance reference [17], but others can be constructed): each one corresponding to a different method of eliminating the spurious degrees of freedom. In this sense the Manifestly Invariant formalism which we have presented here is less "intrinsic" that the Manifestly Predictive one. Nevertheless it leads often to simpler calculations and has a rich inner structure which makes this formalism very useful. We shall use it to define the concept of the P.I.S. associated with the hereditary dynamical system (3.13). We shall say that a P.I.S.

$$\frac{du^\alpha_a}{d\tau} = \xi^\alpha_a\,(x^\beta_b,\,u^\gamma_c\,;\,e_d) \qquad (4.14)$$

where the functions ξ^α_a depend on the charges e_a, is the P.I.S. associated with the dynamical system (3.13) if i) it is one of its reductions, i.e., all the solutions of eqs. (4.14) are solutions of eqs. (3.13) and ii) the functions ξ^α_a can be developped as power series of the charges of the following type:

$$\xi^\alpha_a = \sum_{r,s=1}^{\infty} \xi^{\alpha(r,s)}_a \qquad (4.15)$$

where $\xi^{\alpha(r,s)}_a$ means a term proportional to $e^r_a\ e^s_{a'}$. The assumption that the terms $\xi^{\alpha(p,q)}_a$ are zero for p or $q = 0$ is essential and it expresses of course that the reduction (4.14) has to be compatible with the Principle of Inertia when at least one of the charges is zero. The usual construction of the series (4.15) up to second order,

i.e., up to terms containing four charges uses the fact [8] that the functions ξ_a^α have to be solutions of the following system of integral equations:

$$\xi_a^\alpha = \tilde{W}_a^{\alpha[2]*} + \hat{\zeta}_{aa'} \int_0^1 d\lambda \, R_{a'}(\lambda \hat{\zeta}_{aa'}) \left(\xi_{a'}^\rho \cdot \frac{\partial \xi_a^\alpha}{\partial u^{a'\rho}} \right) \tag{4.16}$$

where: i)

$$\hat{\zeta}_{aa'} = -(x_{aa'} u_{a'}) - r_{aa'}, \quad x_{aa'}^\alpha = x_a^\alpha - x_{a'}^\alpha, \quad r_{aa'}^2 = x_{aa'}^2 + (x_{aa'} u_{a'})^2 \tag{4.17}$$

ii) $R_a(\mu)$ is a shift operator which acts on the functions as follows:

$$R_{a'}(\mu) f(x_a^\alpha, x_a^\beta, u_b^\gamma) = f(x_a^\alpha, x_{a'}^\beta + \mu u_{a'}^\beta, u_b^\gamma) \tag{4.18}$$

and iii) $\tilde{W}_a^{\alpha[2]*}$ is the function (3.12) with the variables $\hat{x}_{aa'}^\alpha$ and $\hat{u}_{aa'}^\alpha$, replaced respectively by $x_{a'}^\alpha + \hat{\zeta}_{aa'} u_{a'}^\alpha$ and $u_{a'}^\alpha$. Since the lowest order of the integrals in eqs. (4.16) is proportional to the product of two charges these equations are the basis of a recurrent algorithm to calculate the series (4.15). The first order approximation is:

$$\xi_a^{\alpha[1]} = e_a e_{a'} r_{aa'}^{-3} \left[-(u_a u_{a'}) x_{aa'}^\alpha + (x_{aa'} u_a) u_{a'}^\alpha \right] \tag{4.19}$$

the explicit expression of the terms $\xi_a^{\alpha(1,1)}$ can be found in [18], [19] and that of $\xi_a^{\alpha(3,1)}$ in [20]. The other terms of order 2 are zero. Since the quantities $W_a^{\alpha[2]}$ are themselves limited to second order it would be meaningless in this case to proceed to the calculation of higher order terms.

Let us assume that the system (4.14) is the P.I.S. associated with the hereditary one (3.13). Considering the solutions of (4.14), the quantities $\hat{x}_{aa'}^\alpha$ and $\hat{u}_{aa'}^\alpha$ defined in paragraph 1 can be considered as functions of (x_a^α, u_b^β). Another construction of the series (4.15), first considered in reference [18], uses the fact that these functions have to satisfy the functional equations:

$$\xi_a^\alpha(x_b^\beta, u_c^\gamma) = \tilde{W}_a^{\alpha[2]} \left[x_{a'}^\beta, u_{a'}^\gamma, \hat{x}_{aa'}^\beta(x,u), \hat{u}_{aa'}^\gamma(x,u) \right] \tag{4.20}$$

and be solutions of the following system of integral equations:

$$\left.\begin{aligned} \hat{x}_{aa'}^\alpha &= x_{a'}^\alpha + \hat{\zeta}_{aa'} u_{a'}^\alpha + \hat{\zeta}_{aa'} \int_0^1 d\lambda \, R_{a'}(\lambda \hat{\zeta}_{aa'}) \left(\xi_{a'}^\rho \cdot \frac{\partial \hat{x}_{aa'}^\alpha}{\partial u^{a'\rho}} \right) \\ \hat{u}_{aa'}^\alpha &= u_{a'}^\alpha + \hat{\zeta}_{aa'} \int_0^1 d\lambda \, R_{a'}(\lambda \hat{\zeta}_{aa'}) \left(\xi_{a'}^\rho \cdot \frac{\partial \hat{u}_{aa'}^\alpha}{\partial u^{a'\rho}} \right) \end{aligned}\right\} \tag{4.21}$$

Since the lowest order of ξ_b^β is proportional to $e_a e_{a'}$ from these

equations we obtain:

$$\hat{x}^{\alpha(0)}_{aa'} = x^{\alpha}_{a'} + \hat{\zeta}_{aa'} u^{\alpha}_{a'} \quad , \quad \hat{u}^{\alpha(0)}_{aa'} = u^{\alpha}_{a'} \tag{4.22}$$

therefore we shall obtain $\xi^{\alpha(1,1)}_{a}$ by substituting in the right-hand-side of eqs. (4.20) $\hat{x}^{\alpha}_{aa}$, and $\hat{u}^{\beta}_{aa'}$, by its lowest order expressions (4.22). This is the same construction which led to (4.19). Expressions (4.19) and (4.22) can now be used to calculate the corresponding expressions at the next order. As we see, this method is closer to the method that we have presented in the first section of this paper in introducing the concept of P.D.E. associated to a hereditary one. We said there that the possibility of pushing the construction would depend on each particular case. Since the construction above does not really depend on the particular form of the functions $\tilde{w}^{\alpha[2]}_{a}$, we see here that for a large class of causal interactions there exists a universal perturbative method to construct their associate P.I.S.

5. Spontaneous Predictivisation

Numerical solutions of the equations of motion of two electric charges, taking into account or not the radiation reaction forces have been obtained and discussed in the one space dimensional case by various authors [21] - [25] . We consider here the problem of integrating the hereditary equations of motion (3.13) using the method of intervals (in this paragraph we shall say interval instead of step. The word step instead will have its usual meaning in numerical integration language) assuming that i) the two charges are equal or opposite and have the same mass m and ii) there exists a frame of reference for which the space trajectories of the initial world-lines Γ_a are such that a) the middle point of the segment joining the two particles is fixed, b) they are symmetric with respect to it, and c) both trajectories lie on a plane, say π .

We have used the distinguished frame of reference of point ii) above, choosing as origin of coordinates the fixed middle point. We shall designate by $\vec{x}$ the two components vector position of one of the particles, say a , at time t on π . Then $-\vec{x}$ will be at any stage of the integration the position at time t of particle a' . Similarly $\vec{u}$ and $-\vec{u}$ will be respectively the space components of the unit four-velocities of particles a and a' at time t . We shall designate by $\hat{\vec{x}}$ and $\hat{t}$ the coordinates of the retarded event on $L_{a'}$ corresponding to an event $(\vec{x},t)$ on L_a . We shall use as unit of length e^2/m where e is the absolute value of the charge of either particle. (We remember that we have taken $c=1$).

Using these notations and conventions the hereditary dynamical system (3.13) can be written as follows:

$$\frac{d\vec{x}}{d\tau} = \vec{u} \quad , \quad \frac{d\vec{u}}{d\tau} = \tilde{\vec{W}}^{[2]} \tag{5.1}$$

with:

$$\tilde{\vec{W}}^{[2]} = \vec{W}^{(1,1)} + \hat{s}^{-2}\left[-G\hat{r}\,\vec{W}^{(1,1)} + \hat{r}^{-1}(\hat{s}^{-1}\hat{\vec{l}} - \hat{\vec{u}})\right]$$
$$-2\hat{r}^{-1}(\hat{k} - \hat{r}\hat{s}^{-1})\vec{W}^{(1,1)} + \frac{2}{3}G\hat{r}^{-3}(\hat{k}\,\vec{u} - \hat{\vec{u}}) \tag{5.2}$$

where G is +1 for equal charges and -1 for opposite ones, and where:

$$\vec{W}^{(1,1)} = \hat{r}^{-3}(\hat{k}\,\hat{\vec{l}} - \hat{s}\,\hat{\vec{u}}) \tag{5.3}$$

with:

$$\vec{l} = \vec{x} - \hat{\vec{x}}\,, \quad \hat{k} = u^0\hat{u}^0 - \vec{u}\,\hat{\vec{u}}\,, \quad u^0 = (1+\vec{u}^2)^{1/2}, \quad \hat{u}^0 = (1+\hat{\vec{u}}^2)^{1/2}$$

$$\hat{r} = \hat{u}^0|\hat{\vec{l}}| - \vec{u}\cdot\hat{\vec{l}}\,, \quad \hat{s} = u^0|\vec{l}| - \vec{u}\cdot\vec{l} \tag{5.4}$$

The time coordinate will be given by the equation:

$$\frac{dt}{d\tau} = u^0 \tag{5.5}$$

To feed the hereditary system (5.1) with initial conditions satisfying the conditions that we stated before we have assumed that during their initial constraint interval each charge had been pulled with a constant force having the direction of the velocity. More precisely we have assumed that during this interval of time the parametric equations $\vec{x}(\tau)$ of particle a were solutions of equations:

$$\frac{d\vec{x}}{d\tau} = \vec{u} \quad , \quad \frac{d\vec{u}}{d\tau} = K\cdot\frac{u^0}{|\vec{u}|}\,\vec{u} \tag{5.6}$$

where K is a constant. Starting with initial conditions $\vec{x}_o$, $\vec{u}_o$ at t=0 we have integrated backwards in time these equations until we have reached the retarded event $(\vec{x}_N, t_N)$ corresponding to the event $(-\vec{x}_o, 0)$ which is the initial position of particle a' . N indicates here the number of steps of the corresponding interval. We have used a variable step size version of the elementary Euler's method. Let $(\vec{x}_B, t_B)$ be the event corresponding to the end of the B-th step. The step size $\Delta\tau_A$ between the events $(\vec{x}_{A+1}, t_{A+1})$ and $(\vec{x}_A, t_A)$ has been chosen to be

$$\Delta\tau_A = \frac{1}{N-A}\left\{-t_A u_A^0 + (\vec{x}_o + \vec{x}_A)\cdot\vec{u}_A - \right.$$

$$- \left[t_A u_A^\circ - (\vec{x}_o + \vec{x}_A)\,\vec{u}_A \right]^2 + t_A^2 - (\vec{x}_o + \vec{x}_A)^2 \Big\} \qquad (5.7)$$

Let us consider the straight world line passing through the event $(-\vec{x}_A,\ t_A)$ in the direction of $(-\vec{u}_A, u_A^\circ)$. $\Delta\tau_A$ is equal to the proper time measured along this straight line between the event $(-\vec{x}_A, t_A)$ and the intersection of it with the null past cone with vertex at the point $(\vec{x}_o, 0)$ divided by N-A . The quantities $\vec{u}_B$ $(B = 0,.., N)$, $\Delta\tau_A$ $(A = 1,..., N)$, $\vec{x}_N$ and t_N from which all relevant information concerning the initial interval can be recovered, were stored for future utilization.

Using the initial conditions Γ_a constructed as we have just mentioned we integrated the hereditary equations (5.1) by the method of intervals using a variable step size version of Euler's method. Let us call $(\vec{x}_{B,J}\ ,\ t_{B,J}\ )$ the coordinates of particle a at the end of the B-th step of the J-th interval. The size of the step $\Delta\tau_{A,I+1}$ between the events $(\vec{x}_{A,I+1},\ t_{A,I+1}\ )$ and $(\vec{x}_{A+1,\ I+1},\ t_{A+1,\ I+1})$ was taken to be:

$$\Delta\tau_{A,I+1} = \hat{k}_{A,I+1}\cdot\Delta\tau_{A,I} - \hat{s}_{A,I+1} + \left[\left(\hat{k}_{A,I+1}\cdot\Delta\tau_{A,I} - \hat{s}_{A,I+1} \right)^2 - \Delta\tau_{A,I}^2 + 2\,\hat{r}_{A,I+1}\cdot\Delta\tau_{A,I} + L_{A,I+1}^2 \right] \qquad (5.8)$$

where:

$$L_{A,I+1}^2 = \hat{\ell}_{A,I+1}^2 - (t_{A,I+1} - t_{A,I})^2 \qquad (5.9)$$

and where $\hat{\ell}_{A,I+1}$, $\hat{k}_{A,I+1}$, $\hat{r}_{A,I+1}$ and $\hat{s}_{A,\ I+1}$ are the quantities defined by eqs. (5.4) corresponding to the event $(\vec{x}_{A,I+1}\ ,\ t_{A,I+1})$. The choice (5.8) guarantees that the event $(\vec{x}_{A+1,\ I+1},\ t_{A+1,\ I+1})$ will be in the null future of the even $(-\vec{x}_{A+1,I},\ t_{A+1,I})$ in the world line of particle a' . L^2 is in principle zero but because of the unaccuracy of the numerical integration, it is not exactly zero. It is therefore necessary to include it in the expression of $\Delta\tau_{A,I+1}$ to prevent a systematic error in the calculation.

At the end of each step we have calculated the acceleration $\vec{A}^{[2]}$ of the corresponding second order associate P.I.S. We have not used the explicit available expressions because they are cumbersome. We have instead proceeded as follows. We have integrated backwards in time, using again the variable step size version of Euler's method, with $\Delta\tau_A$ given by eq. (5.7) , the first order P.I.S. associated with the hereditary equations (5.1). Taking into account eqs. (4.19) this system can be written:

$$\frac{d\vec{u}}{d\tau} = \frac{G}{4}\left[\vec{x}^2 + (\vec{x}\cdot\vec{u})^2\right]^{-3/2}\cdot\left[(1+2\vec{u}^2)\,\vec{x} - (\vec{x}\cdot\vec{u})\vec{u}\right] \qquad (5.10)$$

Starting with initial conditions $(\vec{x}_t, t)$ we have integrated these equations until we reached the event $(\vec{x}_N, t_N)$ which is in the null past of the event $(-\vec{x}_t, t)$. We have then used these data and eqs. (5.2) to calculate the numerical value of $\vec{A}^{[2]}$. These quantities should coincide with the numerical values obtained from the explicit cumbersome expressions modulo third order corrections.

We have considered the two following quantities:

$$D = \frac{|\vec{W}^{[2]}| - |\vec{A}^{[2]}|}{|\vec{W}^{[2]}|}\cdot 100 \quad , \quad S = \left[1 - \frac{\vec{W}^{[2]}\cdot\vec{A}^{[2]}}{|\vec{W}^{[2]}|\cdot|\vec{A}^{[2]}|}\right]^{1/2} \qquad (5.11)$$

which give respectively the percentage difference between the modulus of $\vec{W}^{[2]}$ and $\vec{A}^{[2]}$, and the sine of the angle between these two vectors. The systematic behaviour that we have observed for a variety of initial conditions and values of K is that these quantities tend to two small values D_∞, S_∞ which depend on each particular case. These values are zero if the distance between the particles increases without limit. They are small but non zero if this distance is bounded from above. This being due in our opinion to the fact that $\vec{A}^{[2]}$ it is only an approximate expression. At the end of the first interval the values of D and S are already quite close to their limits. This is connected with the disappearance of the discontinuities of the accelerations at the end points of the first interval. We give below the numerical data that we have obtained in one particular case ($G = -1$) which illustrates these remarks. The initial conditions are: $x=0$, $y=6$, $u_x=0.25$, $u_y=0$, and the value of $K=-0.2$. The number of steps of each interval is 8 (we have used a 319 registers version of an H.P. 41C).

n (steps)	x	y	D	S
2	0.38	5.99	27.3	0.3
4	1.05	5.95	46.6	0.3
6	1.95	5.84	34.0	0.3
8	2.76	5.69	- 3.7	$1.1\ 10^{-2}$
10	3.11	5.59	- 3.6	$1.1\ 10^{-2}$
12	3.73	5.39	- 2.9	$9.4\ 10^{-3}$
14	4.55	5.03	- 1.4	$4.8\ 10^{-3}$
16	5.27	4.63	- 0.8	$1.4\ 10^{-3}$
18	5.58	4.43	- 0.8	$1.3\ 10^{-3}$

The behaviour of the quantities D and S becomes quite different if the distance between the charges becomes too small. This, in our opinion, comes again from the fact that $\vec{A}^{[2]}$ are only approximate quantities unreliable at small distances.

We conclude from all this work that the electromagnetic corresponding to interaction becomes spontaneously predictive, i.e., the P.I.S. $\vec{A}^{[2]}$ is an Attractor of the hereditary system (5.2) as long as the motion of the charges remains confined in a domain of configuration space for which the distance of the charges is not too small. Our numerical exploration would have to be extended and improved to make more precise statements in particular abouth the border limiting the domain beyond which there is no spontaneous predictivisation.

We have done a similar work with the gravitational interaction and we have reached similar conclusions. Spontaneous Predictivisation is therefore a mechanism which works for the very simple examples that we have considered in the first section and for the Electromagnetic and Gravitational interactions. Would it be too rash to conjecture [26] that it is a universal mechanism connected with pure retarded equations?

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26) In the meantime between the workshop and the publication of these proceedings, that conjecture has been proved.

FORMS OF RELATIVISTIC QUANTUM DYNAMICS
(Particles vs. Fields)

F. Coester
Argonne National Laboratory+
Argonne, IL 60439

A systematic presentation of relativistic quantum mechanics usually begins by specifying the properties of an algebra of operators called "observables". It is then assumed that the principles of relativity require that this algebra be generated by local fields. Indeed, theories in which the algebra of dynamical variables is generated by the canonical coordinates and momenta of particles lead to unacceptable conclusions if one assumes that the canonical coordinates are observable particle positions[1]. That assumption is not necessary. In the absence of long-range external fields, only asymptotically free particles are actually observable. The results of a scattering theory are the observables. The general framework of relativistic scattering theory can accommodate either elementary fields or elementary particles. My aim in this lecture is to contrast relativistic field theories and particle theories and to show how particle dynamics can be constructed in agreement with the requirement of cluster separability[2].

It will be convenient to introduce the general framework and the notation by reviewing the assumptions and some results of an abstract scalar field theory[3,4], without emphasis on mathematical rigor[5].

1. The Hilbert Space

The states ψ are vectors in a Hilbert space $\mathcal{H}$.

2. The Field

All dynamical variables are functionals of a local scalar field $A(x)$, $x = \{\vec{x},t\}$; the fields commute for space-like separations.

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$$[A(x), A(x')] = 0 \tag{1}$$

for $(x-x')^2 = (\vec{x}-\vec{x}')^2 - (t-t')^2 > 0$.

3. Poincaré Invariance

The relativistic transformation law of the states is given by a continuous unitary representation $U(d,\Lambda)$ of the Poincaré group, and the field $A(x)$ satisfies the covariance relation

$$U(d,\Lambda)\cdot A(x)\cdot U^{-1}(d,\Lambda) = A(\Lambda x+d) \tag{2}$$

The generators of the infinitesimal transformations are $\vec{P},H$ for the space and time translation, $\vec{J},\vec{K}$ for the rotations and Lorentz boosts. The set of all ten generators is denoted by G ,

$$G = \{\vec{P}, H, \vec{J}, \vec{K}\} \tag{3}$$

The generators satisfy the commutation relations

$$[P_i, P_j] = 0 \quad , \quad [\vec{P}, H] = 0 \tag{4}$$

$$[J_i, J_k] = i\sum_m \epsilon_{ikm} J_m \tag{5}$$

$$[J_i, P_k] = i\sum_m \epsilon_{ikm} P_m \quad , \quad [\vec{J}, H] = 0 \tag{6}$$

$$[J_i, K_k] = i\sum_m \epsilon_{ikm} K_m \tag{7}$$

$$[K_i, K_k] = -i\sum_m \epsilon_{ikm} J_m \tag{8}$$

$$[K_i, P_k] = i\,\delta_{ik} H \quad , \quad [\vec{K}, H] = i\vec{P} \tag{9}$$

It follows that $\vec{P},H$ and $\vec{J},\vec{K}$ transform respectively as a four vector $\{P^\mu\}$ and as an antisymmetric tensor $\{J^{\mu\nu}\}$, where $J^{0i} = K_i$ and $J^{12} = J_3$.

4. The Vacuum

There is a unique invariant vacuum state $|0\rangle$, $U(d,\Lambda)\,|0\rangle = |0\rangle$.

5. The Physical Interpretation

The operators $\vec{J},\vec{P}$ and H are interpreted respectively as the angular momentum, the momentum and the energy of the system. The ope-

rator $M^2 := H^2 - \vec{P}^2$ is the square of the mass. The spectra of H and M are nonnegative and the vacuum state is the only state on which M vanishes, this means we exclude zero-mass particles. Since H governs the time evolution the dynamics of the system is determined if H is known as a function (or functional) of the elementary dynamical variables.

6. One-Particle States

In the orthogonal complement of the vacuum state the mass operator M has a point spectrum

$$0 < m_1 \leq m_2 \leq m_3 \quad \dots\dots ,$$

and a continuous spectrum from $2m_1$ to ∞. To each eigenvalue m_i belongs to an invariant subspace $\mathcal{H}_1^{(i)} \subset \mathcal{H}$ on which $U(d,\Lambda)$ is an irreducible representation belonging to the mass m_i and the spin s_i.

For the sake of simplicity we assume in the following that there is only one mass eigenvalue m and that the spin vanishes. A one-particle state $\psi_1 \in \mathcal{H}_1$ can be expressed in the form

$$\psi_1 = \int d^3p \, |\vec{p}\rangle \chi(\vec{p}) \tag{10}$$

where $\chi(\vec{p}) \in \mathcal{L}^2$ and $|\vec{p}\rangle$ transforms under Poincaré transformations according to

$$e^{i P\cdot d} |\vec{p}\rangle = |\vec{p}\rangle \, e^{i(\vec{p}\cdot\vec{d} - \omega d^0)} \tag{11}$$

and

$$U(\Lambda) |\vec{p}\rangle = |\vec{\hat{p}}\rangle \cdot \sqrt{\hat{\omega}/\omega} \tag{12}$$

where $\omega := \sqrt{\vec{p}^2 + m^2}$, $\hat{p} := \Lambda p$, $p := \{\vec{p}, \omega\}$, $U(\Lambda) := U(0,\Lambda)$.

Let $A(f,t)$ be defined by

$$A(f,t) := \frac{i}{\hbar} \int d^3x \left\{ A(x) \frac{\partial f}{\partial t} - \frac{\partial A}{\partial t} \cdot f(x) \right\} \tag{14}$$

where

$$f(x) := \int d^3p \cdot \omega^{-1/2} \cdot \chi(\vec{p}) \cdot e^{i(\vec{p}\cdot\vec{x} - \omega t)} \tag{15}$$

and $\chi(\vec{p})$ is a smooth function. We assume that the matrix elements

$\langle \vec{p} |\, A(f,t)\, | 0 \rangle$ do not vanish. It follows from the covariance of the field $A(x)$ that a constant factor can be chosen such that

$$\langle \vec{p} |\, A(f,t)\, | 0 \rangle = \chi(\vec{p}) \tag{16}$$

It is then possible to construct a covariant field $B(x)$ as a linear functional of $A(x)$ such that[6]

$$B(f,t)\,|0\rangle = \int d^3p \cdot |\vec{p}\rangle\, \chi(\vec{p}) \tag{17}$$

7. Scattering States . The states

$$\phi(t) := B(f_1,t) \cdot B(f_2,t) \ldots B(f_n,t)\,|0\rangle \tag{18}$$

satisfy

$$\| \phi(t_1) - \phi(t_2) \| \longrightarrow 0 \tag{19}$$

as $t_1, t_2 \longrightarrow \pm \infty$.
They have therefore strong limits

$$\underset{t \to \pm\infty}{s\text{-}\lim} \phi(t) = \psi^{(\pm)} \tag{20}$$

The states $\psi^{(\pm)}$ are the scattering states and the S matrix is

$$S_{\beta\alpha} = (\psi_\beta^{(+)} , \psi_\alpha^{(-)}) \tag{21}$$

The vector $\phi(t)$ defined by (18) is a linear functional of the tensor product $\bigotimes_{i=1}^{n} \chi_i$, i.e.

$$\phi(t) = \Phi(t) \bigotimes_{i=1}^{n} \chi_i \tag{22}$$

If we define the operator H_f by

$$H_f \bigotimes_{i=1}^{n} \chi_i = \Big(\sum_{i=1}^{n} \omega_i \Big) \bigotimes_{i=1}^{n} \chi_i \tag{23}$$

it follows from

$$e^{iHt}\, B(\vec{x},0)\, e^{-iHt} = B(\vec{x},t) \tag{24}$$

that the time dependence of $\Phi(t)$ is given by

$$\Phi(t) = e^{iHt}\,\Phi\, e^{-iH_f t} \tag{25}$$

Let

$$\mathcal{H}_{fn} := \bigotimes_{i=1}^{n} \mathcal{L}^2(\vec{p}_i, d^3p_i) \tag{26}$$

and

$$\mathcal{H}_f := \bigoplus_{n=1}^{\infty} \mathcal{H}_{fn} \tag{27}$$

The vectors $B(f,0)\ldots. |0\rangle$ define the operator Φ from $\mathcal{H}_f$ into $\mathcal{H}$ and the scattering states $\Psi^{(\pm)}$ are

$$\Psi^{(\pm)} = \Omega_{\pm}\chi \tag{28}$$

with $\chi \in \mathcal{H}_f$ and the operators $\Omega_{\pm}$,

$$\Omega_{\pm} := \underset{t\to\pm\infty}{s\text{-}\lim}\; e^{iHt}\,\Phi\, e^{-iH_f t} \tag{29}$$

are generalized wave operators[7]. The assumption of asymptotic completeness can be stated in the form

$$\Omega_+\Omega_+^{\dagger} = \Omega_-\Omega_-^{\dagger} = 1 \tag{30}$$

It follows that the S operator,

$$S := \Omega_+^{\dagger}\Omega_- \tag{31}$$

is a unitary operator in $\mathcal{H}_f$.

Let G_{fi} be the generators of the irreducible representation of the Poincaré group on $\mathcal{H}_{fi}$ belonging to the mass m and spin zero. The generators of the Poincaré transformations of $\mathcal{H}_f$ are given by

$$G_f = \sum_i G_{fi} \tag{32}$$

The Poincaré invariance of the scattering states is expressed by the intertwining relations

$$G\Omega_{\pm} = \Omega_{\pm} G_f \tag{33}$$

The invariance of the S operator

$$[G_f, S] = 0 \tag{34}$$

is a consequence. The proof of the existence and invariance of the scattering states[3,8] depends critically on the assumed properties of the local field $A(x)$. Note that our construction of the operator Φ involves an integral over the hypersurface $t=0$ in the definition $B(f,0)$. Thus Φ is invariant under translations and rotations

$$\vec{P}\,\Phi = \Phi P_f \quad , \quad \vec{J}\,\Phi = \Phi J_f \tag{35}$$

but not under Lorentz boosts;

$$\vec{K}\,\Phi \neq \Phi \vec{K}_f \tag{36}$$

We have thus an "instant-form"[9] dynamics. Had we integrated over an invariant hyperboloid or the hyperplane $x_3 + t=0$ we would have obtained a "point form" or a "front form".

If we have Poincaré generators in $\mathcal{H}$ for a noninteracting system and add interaction terms to the Hamiltonian then, according to (9), either $\vec{K}$ or $\vec{P}$, or both, must also be interaction dependent. In canonical field theories the Poincaré generators are expressed as integrals over the energy-momentum tensor $T^{\mu\nu}(x)$

$$H = \int d^3x \; T^{00}(\vec{x}) \tag{37}$$

$$P^i = \int d^3x \; T^{0i}(\vec{x}) \tag{38}$$

$$\vec{K} = \int d^3x \cdot \vec{x} \cdot T^{00}(\vec{x}) \tag{39}$$

$$J_i = \frac{1}{2} \sum_{m,n} \varepsilon_{imn} \int d^3x \left(x^m T^{0n}(\vec{x}) - x^n T^{0m}(\vec{x}) \right) \tag{40}$$

This construction solves the problem of finding compatible interaction dependences for H and $\vec{K}$. The commutation relations (4)-(9) are satisfied if and only if the energy density $T^{00}(\vec{x})$ satisfies the local Schwinger[10] commutation relations

$$\begin{aligned} i[T^{00}(\vec{x}), T^{00}(\vec{x}')] = \sum_\kappa \{ & T^{0\kappa}(\vec{x}) \cdot \partial_\kappa \delta(\vec{x}-\vec{x}') - \\ & - T^{0\kappa}(\vec{x}') \cdot \partial'_\kappa \delta(\vec{x}-\vec{x}') \} + \sigma(\vec{x},\vec{x}') \end{aligned} \tag{41}$$

where the function $\sigma(\vec{x},\vec{x}')$ must be antisymmetric, $\sigma(\vec{x},\vec{x}') = -\sigma(\vec{x}',\vec{x})$ and satisfy the relations,

$$\int d^3x \cdot \sigma(\vec{x},\vec{x}') = 0$$
$$\int d^3x \cdot \vec{x} \cdot \sigma(\vec{x},\vec{x}') = 0 \qquad (42)$$

The condition $\sigma = 0$ is sufficient but not necessary. Clearly the locality features of the field theory are essential for the relativistic invariance in this construction. Particle creation and the necessity for infinitely many degrees of freedom are thus intimately connected with the relativistic invariance.

The question remains whether a satisfactory Poincaré representation can be constructed if the elementary dynamical variables are the canonical coordinates, momenta and spins of a finite number of particles. In the absence of the locality features of field theories, it seems reasonable to impose the following cluster separability requirement[2]. Let a denote a partition of the N-particle system into disjoint clusters a_i, $i=1 \ldots n_a$. The states of the cluster a_i are vectors in a Hilbert space $\mathcal{H}_{ai}$. The Hilbert space $\mathcal{H}$ is the N fold tensor product of one-particle spaces and hence

$$\mathcal{H} = \bigotimes_{i=1}^{n_a} \mathcal{H}_{ai} \qquad (43)$$

for every partition a. Let $U_{ai}(d,\Lambda)$ be a unitary Poincaré representation for the cluster a_i. We will also use the notation $U_{ai}(d,\Lambda)$ for the operator $U_{ai}(d,\Lambda) \otimes 1$ acting on the tensor-product space (43). The representation $U_a(d,\Lambda)$ describing the noninteracting clusters of the partition a is then given by

$$U_a(d,\Lambda) = \prod_{i=1}^{n_a} U_{ai}(d,\Lambda) \qquad (44)$$

Equation (44) implies the relation

$$G_a = \sum_{i=1}^{n_a} G_{ai} \qquad (45)$$

for the generators. The operator $T_a(\underset{\sim}{d})$, $\underset{\sim}{d} = d_1, d_2, \ldots, d_{na}$,

$$T_a(\underset{\sim}{d}) := \prod_{i=1}^{n_a} U_{ai}(d_i,1) \qquad (46)$$

translates the clusters of the partition relative to each other. The cluster separability requirement for the representation $U(d,\Lambda)$ of

the complete system is then

$$\underset{\min(a_i-a_j)^2\to\infty}{\text{s-lim}} \left(U(a,\Lambda) - U_0(a,\Lambda)\right) T_0(a) = 0 \tag{47}$$

Several functions of the generators will play an essential role in the further development. The covariant spin vector W_μ is defined by

$$W_\mu := \frac{1}{2} J^{\rho\sigma} P^\nu \varepsilon_{\nu\rho\sigma\mu} \tag{48}$$

It follows that

$$W^0 = \vec{P}\cdot\vec{J} \quad , \quad \vec{W} = H\vec{J} + \vec{P}\times\vec{K} \tag{49}$$

The Newton-Wigner position operator[11] can be defined as a function of the generators by

$$\vec{X} := \frac{1}{2}\left\{H^{-1}\vec{K} + \vec{K}H^{-1}\right\} - \frac{\vec{P}\times\vec{W}}{MH(M+H)} \tag{50}$$

It follows from this definition and the commutation relations (4)-(9) that $\vec{X}$ and $\vec{P}$ satisfy canonical commutation rules,

$$[X_i, X_j] = 0 \quad , \quad [X_i, P_j] = i\delta_{ij} \tag{51}$$

and that they commute with the canonical spin $\vec{j}$ defined by

$$\vec{j} := \vec{J} - \vec{X}\times\vec{P} \tag{52}$$

The canonical spin $\vec{j}$ is related to the covariant spin vector W_μ by a Lorentz transformation

$$L(\vec{Q})W = \{M\vec{j}, 0\} \tag{53}$$

where $\vec{Q} := \vec{P}/M$ and $L(\vec{Q})$ is the inverse boost defined by

$$L(\vec{Q})\left\{\vec{Q}, \sqrt{1+\vec{Q}^2}\right\} = (0,0,0,1) \tag{54}$$

From Eq. (53) it follows that

$$W^\mu W_\mu = M^2\vec{j}^{\,2} \tag{55}$$

Conversely the generators can be expressed as functions of the operators $M, \vec{P}, \vec{X}$ and $\vec{j}$ defined to satisfy the commutation relations (51) and

$$[\vec{j}, M] = [\vec{X}, M] = [\vec{P}, M] = 0 \tag{56}$$

$$[\vec{j}_i, \vec{j}_k] = i \sum_m \epsilon_{ikm} \vec{j}_m \tag{57}$$

$$[\vec{j}, \vec{X}] = [\vec{j}, \vec{P}] = 0 \tag{58}$$

We then have the expressions

$$H = \sqrt{\vec{P}^2 + M^2} \tag{59}$$

$$\vec{J} = \vec{X} \times \vec{P} + \vec{j} \tag{60}$$

and

$$\vec{K} = \frac{1}{2} \{ H\vec{X} + \vec{X} H \} - (\vec{j} \times \vec{P}) (M+H)^{-1} \tag{61}$$

for the generators which we write schematically as

$$G = G_I (M, \vec{X}, \vec{P}, \vec{J}) \tag{62}$$

In an instant-form dynamics the operators $\vec{P}$ and $\vec{J}$ are the same for interacting and noninteracting particles. The generators for the noninteracting system are

$$G_0 = G_I (M_0, \vec{X}_0, \vec{P}, \vec{J}) \tag{63}$$

The Bakamjian-Thomas[12] construction for interacting particles is then

$$G_{BT} = G_I (M_0 + v, \vec{X}_0, \vec{P}, \vec{J}) \tag{64}$$

where v commutes with $\vec{P}$, $\vec{J}$ and $\vec{X}_0$.

Similar constructions are possible in point-form and front-form dynamics[13]. We may scale $\vec{X}$ and $\vec{P}$ by M and define

$$\vec{Q} := \vec{P}/M \quad , \quad \vec{R} := M\vec{X} \quad , \quad \varepsilon := \sqrt{1 + \vec{Q}^2} \tag{65}$$

It follows from (54)-(61) that

$$H = M\mathcal{E} \quad , \quad \vec{P} = M\vec{\mathcal{Q}} \tag{66}$$

$$\vec{J} = \vec{\mathcal{R}} \times \vec{\mathcal{Q}} + \vec{j} \tag{67}$$

and

$$\vec{\mathcal{K}} = \frac{1}{\mathcal{E}}(\vec{\mathcal{R}}\mathcal{E} + \mathcal{E}\vec{\mathcal{R}}) - \vec{j} \times \vec{\mathcal{Q}}\,(1+\mathcal{E})^{-1} \tag{68}$$

since $\vec{R}$ can be expressed as a function of $\vec{J},\vec{K}$ and $\vec{Q}$ we can write (66)-(68) schematically as

$$G = G_P(M, \vec{\mathcal{Q}}, \vec{J}, \vec{\mathcal{K}}) \tag{69}$$

and the Bakamjian-Thomas construction is[14,15]

$$G_{BT} = G_P(M_0+v, \vec{\mathcal{Q}}_0, \vec{J}, \vec{\mathcal{K}}) \tag{70}$$

where v commutes with $\vec{J},\vec{K}$ and $\vec{Q}_0$, and $\vec{J},\vec{K}$ are the same for the interacting and the noninteracting system.

For the front form we need to identify the generators that leave the null planes $\vec{n}.\vec{x}+t$ = const. invariant. They are $\vec{n}.\vec{K}$, $\vec{n}.\vec{J}$ and

$$P_+ := H + \vec{n}\cdot\vec{P} \quad , \quad \vec{P}_\perp := \vec{P} - \vec{n}\,(\vec{n}\cdot\vec{P}) \tag{71}$$

$$\vec{E} := \vec{K}_\perp + \vec{n} \times \vec{J} \tag{72}$$

where

$$\vec{K}_\perp := \vec{K} - \vec{n}\,(\vec{n}\cdot\vec{K}) \tag{73}$$

In the front form the interaction dependent generators are

$$P_- := H - \vec{n}\cdot\vec{P} \tag{74}$$

and

$$\vec{F} := \vec{K}_\perp - \vec{n} \times \vec{J} \tag{75}$$

The front-form Bakamjian-Thomas construction is schematically[13]

$$G_{BT} = G_F(M_0+v,\ \vec{s}_0,\ P_+, \vec{P}_\perp, \vec{E}, \vec{n}\cdot\vec{K}, \vec{n}\cdot\vec{J})$$

where v commutes with P_+, $\vec{P}_\perp$, $\vec{E}$, $\vec{n}.\vec{K}$, $\vec{n}.\vec{J}$ and $\vec{s}_0$, and $\vec{s}$ is a spin vector that satisfies

$$\vec{S}^2 = \vec{j}^2 \tag{76}$$

In formulating the scattering theory we concentrate again on the instant form. Let a be a partition of the particles into n_a distinct clusters such that for each cluster a_i the mass operator M_{ai} has at least one point eigenvalue. The corresponding eigenvectors $|\alpha_i, \vec{p}, \mu\rangle$ define a channel α. They satisfy

$$M_{ai} |\alpha_i, \vec{p}, \mu\rangle = |\alpha_i, \vec{p}, \mu\rangle m_{\alpha i} \tag{77}$$

$$\vec{j}^2_{ai} |\alpha_i, \vec{p}, \mu\rangle = |\alpha_i, \vec{p}, \mu\rangle S_{\alpha i}(S_{\alpha i}+1) \tag{78}$$

$$\vec{P}_{ai} |\alpha_i, \vec{p}, \mu\rangle = |\alpha_i, \vec{p}, \mu\rangle \vec{p} \tag{79}$$

and transform under Lorentz transformations according to

$$U_{ai}(\Lambda) |\alpha_i, \vec{p}, \mu\rangle = \sum_{\mu'} |\alpha_i, \hat{p}, \mu'\rangle D^{s_{ai}}_{\mu'\mu}[R(\Lambda, \hat{p})] \cdot \sqrt{\hat{\omega}/\omega} \tag{80}$$

where D^s is the $2s+1$ dimensional irreducible representation of the Wigner rotation defined as the product of three Lorentz transformations

$$R(\Lambda, \hat{p}) = L(\hat{p}/m)\, \Lambda\, L^{-1}(\hat{p}/m) \tag{81}$$

The tensor product of these eigenfunctions defines the operator Φ_α from $\mathcal{H}_{f\alpha}$ into $\mathcal{H}$. With

$$\mathcal{H}_f = \bigoplus_\alpha \mathcal{H}_{f\alpha} \tag{82}$$

and

$$\Phi_\alpha \mathcal{H}_{f\beta} = 0 \qquad \text{for} \qquad \alpha \neq \beta \tag{83}$$

we define

$$\Phi := \sum_\alpha \Phi_\alpha \tag{84}$$

This injection operator Φ has the same symmetry properties (35), (36) as before. Existence of the wave operators

$$\Omega_\pm(H, \Phi, H_f) := \underset{t \to \pm\infty}{s\text{-}\lim} e^{iHt} \Phi\, e^{-iH_f t} \tag{85}$$

does not guarantee their Lorentz invariance[16].

In order to discuss the existence and invariance of the wave operators it is useful to note the tensor product structure

$$\mathcal{H} = \mathcal{L}^2(\vec{P}, d^3\vec{P}) \otimes \hat{\mathcal{H}} \tag{86}$$

and

$$\mathcal{H}_f = \mathcal{L}^2(\vec{P}, d^3\vec{P}) \otimes \hat{\mathcal{H}}_f \tag{87}$$

Any vector $\psi \in \mathcal{H}$ is represented by a vector valued function $\hat{\psi}(\vec{P}) \in \hat{\mathcal{H}}$ such that

$$\|\psi\|^2 = \int d^3P \cdot \|\hat{\psi}(\vec{P})\|^2 \tag{88}$$

Any translationally invariant operator $\mathcal{O}$ has the representation

$$(\vec{P}' | \mathcal{O} | \vec{P}) = \delta(\vec{P}' - \vec{P})\, \hat{\mathcal{O}}(\vec{P}) \tag{89}$$

where $\hat{\mathcal{O}}(\vec{P})$ operates on $\hat{\mathcal{H}}$. For the injection operator Φ we have

$$(\vec{P} | \Phi | \vec{P}_f) = \delta(\vec{P} - \vec{P}_f)\, \hat{\Phi}(\vec{P}) \tag{90}$$

where $\hat{\Phi}(\vec{P})$ maps $\mathcal{H}_f$ into $\mathcal{H}$. It follows that if $\mathcal{V}$ is defined by

$$\mathcal{V} := H\Phi - \Phi H_f \tag{91}$$

then

$$\hat{\mathcal{V}}(\vec{P}) = \hat{H}(\vec{P})\, \hat{\Phi}(\vec{P}) - \hat{\Phi}(\vec{P})\, \hat{H}_f(\vec{P}) \tag{92}$$

We are now in a position to state sufficent conditions for the existence of the wave operators $\Omega_\pm$.

Theorem 1[17]. If for every momentum $\vec{P}$ there is a dense set $D \subset D(\hat{H}_f)$ such that for $\hat{\chi} \in D$

$$\hat{\Phi}\, e^{\pm i \hat{H}_f t}\, \hat{\chi} \in D(\hat{H}) \cap \hat{\Phi}\, D(\hat{H}_f)$$

$e^{\pm i H_f t}\, \hat{\chi}$ is strongly continuous in t and

$$\int_0^\infty dt \; \| \hat{V}(\vec{P}) \, e^{\pm i \hat{H}_f t} \hat{\chi} \| < \infty$$

then

$$\underset{t \to \pm\infty}{\text{s-lim}} \; e^{i \hat{H}(\vec{P}) t} \cdot \hat{\Phi}(\vec{P}) \cdot e^{-i \hat{H}_f(\vec{P}) t} = \hat{\Omega}_\pm(\vec{P})$$

exists.

From theorem 1 it follows that for any $\chi \in \mathcal{H}_f$ there is some constant C independent of t and $\vec{P}$ such that

$$\| \{ e^{\pm i \hat{H}(\vec{P}) t} \hat{\Phi}(\vec{P}) \, e^{\pm i \hat{H}_f(\vec{P}) t} - \Omega_\pm(\vec{P}) \} \hat{\chi}(\vec{P}) \|^2 < C \| \hat{\chi}(\vec{P}) \|^2 \tag{93}$$

It follows therefore from the dominated convergence theorem[18] that

$$\lim_{t \to \pm\infty} \int d^3p \, \| \{ e^{i \hat{H}(\vec{P}) t} \cdot \hat{\Phi}(\vec{P}) \cdot e^{-i \hat{H}_f(\vec{P}) t} - \Omega_\pm(\vec{P}) \} \hat{\chi}(\vec{P}) \|^2 = 0 \tag{94}$$

and hence

$$(\vec{P} \,|\, \Omega_\pm(H, \Phi, H_f) \,|\, \vec{P}_f) = \delta(\vec{P} - \vec{P}_f) \, \hat{\Omega}_\pm(\vec{P}) \tag{95}$$

Theorem 2. If either H, Φ or M, Φ satisfy the conditions of theorem 1 then both $\Omega_\pm(H, \Phi, H_f)$ and $\Omega_\pm(M, \Phi, M_f)$ exist and they are equal.

From the generalized Kato-Birman invariance principle[19] and theorem 1 it follows that both $\Omega_\pm(\hat{H}, \hat{\Phi}, \hat{H}_f)$ and $\Omega_\pm(\hat{M}, \hat{\Phi}, \hat{M}_f)$ exist and that they are equal,

$$\Omega_\pm(\hat{H}, \hat{\Phi}, \hat{H}_f) = \Omega_\pm(\hat{M}, \hat{\Phi}, \hat{M}_f) = \hat{\Omega}_\pm \tag{96}$$

In the following we assume that the conditions of theorem 1 are satisfied. If the injection operator Φ satisfies the condition

$$\vec{X} \Phi = \Phi \hat{X}_f \tag{97}$$

then

$$\vec{X} \Omega_\pm = \Omega_\pm \vec{X}_f$$

It follows that the wave operators $\Omega_\pm$ and the S operator are Lorentz invariant. Conversely if $\Omega_\pm(H, \Phi, H_f)$ is Lorentz invariant then there exists a Φ' such that[13]

$$\Omega_\pm(H, \Phi, H_f) = \Omega_\pm(H, \Phi', H_f) \tag{98}$$

and

$$\vec{X}\Phi' = \Phi'\vec{X}_f \tag{99}$$

The construction of interacting representations that satisfy cluster separability proceeds inductively for an increasing number of particles. For two particles the Bakamjian-Thomas construction satisfies cluster separability and $\vec{X}_0\Phi = \Phi\vec{X}_f$. For three particles Mutze[20] has shown that the Bakamjian-Thomas construction cannot satisfy cluster separability unless all two-body interactions vanish. However, for two interacting particles and a noninteracting spectator it is manifestly possible to construct a representation G and Lorentz invariant wave operators which satisfy cluster separability while $\vec{X} \neq \vec{X}_0$.

The recursive construction for a fully interactive N-particle system proceeds along the following lines[13,21]. Suppose the problem is solved for $N' < N$ then we have for all partitions a into n_a clusters, $n_a > 1$, a representation $G(a)$ of the form

$$G(a) = \sum_{i=1}^{n_a} G_{ai} \tag{100}$$

These generators satisfy by assumption the cluster separability condition

$$(G(a))_b = G(a \cap b) \tag{101}$$

where, for any operator $\mathcal{O}$, $\mathcal{O}_a$ is the operator obtained from $\mathcal{O}$ by turning off all interactions between different clusters of a . Knowing $G(a)$ for all a is sufficient for the construction of the injection operator Φ and a Euclidean invariant unitary operator $A(a)$ which satisfies

$$A(a)\vec{X}_a = \vec{X}_0 A(a) \tag{102}$$

and

$$(A(a))_b = A(a \cap b) \tag{103}$$

The desired generators for the fully interacting N-particle system are then

$$G = A^{-1} G_I(\tilde{M}, \vec{X}_0, \vec{P}, \vec{J})\, A \tag{104}$$

where

$$A := \exp\left(\sum_a C_a \ln A(a)\right) \tag{105}$$

$$C_a := (-1)^{n_a}(n_a-1)! \tag{106}$$

and

$$\tilde{M} := \sum_a C_a A_a M_a A_a^{-1} \tag{107}$$

Note that

$$A_a = A(a) \quad \text{and} \quad G_a = G(a) \tag{108}$$

From Φ we can construct an injection operator $\bar{\Phi}$ satisfying

$$\vec{X}_o \bar{\Phi} = \bar{\Phi} \vec{X}_f \tag{109}$$

such that

$$s\text{-}\lim_{t\to\pm\infty} \left(\Phi - A^{-1}\bar{\Phi}\right) e^{iH_f t} = 0 \tag{110}$$

It follows that

$$\Omega_\pm(\tilde{M}, \Phi, M_f) = A\, \Omega_\pm(H, \Phi, H_f) \tag{111}$$

and hence the wave operators $\Omega_\pm(H, \Phi, H_f)$ are Lorentz invariant,

$$\vec{\kappa}\, \Omega_\pm(H, \Phi, H_f) = \Omega_\pm(H, \Phi, H_f)\, \vec{\kappa}_f \tag{112}$$

For interesting applications particle creation is clearly essential, but it is not required by relativistic invariance. It is worth noting that particle theories do not become field theories when they are generalized to include particle creation. The relativistic Lee model[22,13] is not a field theory. The elementary particles are N, θ and V. The $N\theta$ system allows a Bakamjian-Thomas construction where v has matrix elements $N\theta \rightleftarrows V$. In many-body systems the numbers N_N+N_V and $N_\theta+N_V$ are conserved. In the recursive construction of the many-body representation these numbers play the same role as the total particle number before.

Perhaps more interesting is an $N\pi$ system with a vertex interaction $N\pi \rightleftarrows N$ in the mass operator. The system can be truncated to allow at most one pion without loosing the relativistic invariance, but at the expense of cluster separability. Cluster separability can be achieved

as the number of pion is allowed to increase indefinitely, but the theory does not become a field theory in that limit.

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RELATIVISTIC-PARTICLE QUANTUM MECHANICS
(Applications and Approximations)

F. Coester
Argonne National Laboratory+
Argonne, IL 60439

In this lecture I hope to show that relativistic-particle quantum mechanics with direct interactions is a useful tool for building models applicable to hadron systems at intermediate energies. To do this I will first describe a class of models designed to incorporate nucleon-nucleon interactions, pion production, absorption and scattering into a single dynamical framework without dressing the nucleons with pion clouds[1,2]. The second major topic concerns electromagnetic interactions. In the first lecture (referred to as I in the following) I specifically excluded long-range forces and zero-mass particles. Since many of the experimental data in hadron physics involve electromagnetic interactions this limitation is a major defect which must be addressed.

The elementary particles of the NNπ model are the nucleon, the isobar and the pion. Let $\mathcal{H}_N$, $\mathcal{H}_\Delta$ and $\mathcal{H}_\pi$ be the Hilbert spaces of the corresponding one-particle states. The Hilbert space of states under consideration is then

$$\mathcal{H} = \mathcal{H}_N \otimes \mathcal{H}_N \oplus \mathcal{H}_N \otimes \mathcal{H}_\Delta \oplus \mathcal{H}_N \otimes \mathcal{H}_N \otimes \mathcal{H}_\pi \qquad (1)$$

The interactions are such that the Δ decays into a pion and a nucleon. The physical particles are the nucleons, the pion and the deuteron. The space is therefore

$$\mathcal{H}_f = \mathcal{H}_{fN} \otimes \mathcal{H}_{fN} \oplus \mathcal{H}_{fd} \otimes \mathcal{H}_\pi \oplus \mathcal{H}_{fN} \otimes \mathcal{H}_{fN} \otimes \mathcal{H}_{f\pi} \qquad (2)$$

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The generators G_o and G_f are defined on these spaces in the obvious manner. Following the general scheme of I we first construct $G_{NN,\pi}$ and $G_{N\pi,N}$ for the partitions $(NN)\pi$ and $(N\pi)N$. Next we need mass operators $\tilde{M}_{NN,\pi}$ and $\tilde{M}_{N\pi,N}$ which commute with $\vec{X}_o$ and are scattering equivalent to $M_{NN,\pi}$ and $M_{N\pi,N}$.

Let $\vec{p}_a$ and $\vec{p}_b$ be the momenta of the two nucleons. States in $\mathcal{H}_{NN} := \mathcal{H}_N \otimes \mathcal{H}_N$ are represented by functions $\psi(\vec{p}_a,\vec{p}_b)$, or equivalently by functions of $\vec{P}_{NN}$ and $\vec{k}_a$, where

$$\vec{P}_{NN} = \vec{p}_a + \vec{p}_b \tag{3}$$

and

$$k_a = L(\vec{P}_{NN}/M^o_{NN})\, p_a \tag{4}$$

Spin variables will be suppressed throughout in order to simplify the notation. The Bakamjian-Thomas construction of M_{NN} is straightforward, i.e.

$$(\vec{P}'_{NN}\ \vec{k}'_a \mid M_{NN} \mid \vec{k}_a\ \vec{P}_{NN}) = \delta(\vec{P}'_{NN}-\vec{P}_{NN})\cdot(\vec{k}'_a \mid \hat{M} \mid \vec{k}_a) \tag{5}$$

where $\hat{M}_{NN}$ is independent of $\vec{P}_{NN}$,

$$(\vec{k}' \mid \hat{M}_{NN} \mid \vec{k}) = 2(\vec{k}^2+m_N^2)^{1/2}\cdot\delta(\vec{k}'-\vec{k}) + (\vec{k}' \mid v_{NN} \mid \vec{k}) \tag{6}$$

The wave operator $\Omega_{NN\pm}$ for nucleon-nucleon scattering is

$$\Omega_{NN\pm} = \Omega_\pm(M_{NN}, M^o_{NN}) = \underset{t\to\pm\infty}{s\text{-}lim}\ e^{iM_{NN}t}\, e^{-iM^o_{NN}t} \tag{7}$$

$$(\vec{P}'_{NN}\ \vec{k}'_a \mid \Omega_{NN\pm} \mid \vec{k}_a\ \vec{P}_{NN}) = \delta(\vec{P}'_{NN}-\vec{P}_{NN})\cdot(\vec{k}'_a \mid \hat{\Omega}_{NN\pm} \mid \vec{k}_a) \tag{8}$$

The wave matrix $\hat{\Omega}_{NN\pm}$ can be obtained from $\hat{v}_{NN}$ by solving a Lippmann-Schwinger equation. In the presence of a pion spectator we have

$$(\vec{p}'_\pi\ \vec{P}'_{NN}\ \vec{k}'_a \mid M_{NN} \mid \vec{k}_a\ \vec{P}_{NN}\ \vec{p}_\pi) = \delta(\vec{p}'_\pi-\vec{p}_\pi)\,(\vec{P}'_{NN}\ \vec{k}'_a \mid M_{NN} \mid \vec{k}_a\ \vec{P}_{NN}) \tag{9}$$

The generators are then additive,

$$H_{NN,\pi} = H_{NN} + H_\pi \tag{10}$$

$$\vec{K}_{NN,\pi} = \vec{K}_{NN} + \vec{K}_{\pi} \tag{11}$$

and the wave operators remain unchanged,

$$\Omega_{NN,\pi\pm} = \Omega_{NN\pm} \tag{12}$$

where

$$(\vec{p}_\pi' \vec{P}_{NN}' \vec{k}_a' | \Omega_{NN\pm} | \vec{k}_a \vec{P}_{NN} \vec{p}_\pi) = \delta(\vec{p}_\pi' - \vec{p}_\pi) \cdot \delta(\vec{P}_{NN}' - \vec{P}_{NN}) \cdot (\vec{k}_a' | \hat{\Omega}_{NN\pm} | \vec{k}_a) \tag{13}$$

Instead of representing states by functions of $\vec{k}_a, \vec{P}_{NN}$ and $\vec{p}_\pi$, we may choose as independent variables $\vec{k}_a, \vec{P}$ and $\vec{q}_\pi$, where $\vec{q}_\pi$ is defined by

$$q_\pi = L(\vec{P}/M_o)\, p_\pi \tag{14}$$

The virtue of this choice is that $\vec{X}_o$ is represented by $i\nabla_P$

$$\vec{X}_o \psi(\vec{k}_a, \vec{P}, \vec{q}_\pi) = i \nabla_P \psi(\vec{k}_a, \vec{P}, \vec{q}_\pi) \tag{15}$$

and that $\tilde{M}_{NN,\pi}$ defined by

$$\tilde{M}_{NN,\pi} := (\vec{q}_\pi^{\,2} + \tilde{M}_{NN}^{\,2})^{1/2} + (\vec{q}_\pi^{\,2} + m_\pi^2)^{1/2} \tag{16}$$

where

$$(\vec{q}_\pi' \vec{P}' \vec{k}_a' | \tilde{M}_{NN} | \vec{k}_a \vec{P} \vec{q}_\pi) = \delta(\vec{q}_\pi' - \vec{q}_\pi) \cdot \delta(\vec{P}' - \vec{P}) \cdot (\vec{k}_a' | \hat{M}_{NN} | \vec{k}_a) \tag{17}$$

commutes with $\vec{X}_o$.

The operators $M_{NN,\pi}$ and $\tilde{M}_{NN,\pi}$ are defined by stipulating that they vanish on $\mathcal{H}_{NN}$ and on $\mathcal{H}_{N\Delta} := \mathcal{H}_N \otimes \mathcal{H}_\Delta$. It follows from (16) and (17) that

$$\tilde{\Omega}_{NN,\pi\pm} = \tilde{\Omega}_{NN\pm} \tag{18}$$

and

$$(\vec{q}_\pi' \vec{P}' \vec{k}_a' | \tilde{\Omega}_{NN\pm} | \vec{k}_a \vec{P} \vec{q}_\pi) = \delta(\vec{q}_\pi' - \vec{q}_\pi) \cdot \delta(\vec{P}' - \vec{P}) (\vec{k}_a' | \hat{\Omega}_{NN\pm} | \vec{k}_a) \tag{19}$$

Since $\vec{q}_\pi' = \vec{q}_\pi$ follows from $\vec{p}_\pi' = \vec{p}_\pi$ and $\vec{P}_{NN}' = \vec{P}_{NN}$ if and

only if $|\vec{k}'_a| = |\vec{k}_a|$, we have

$$S_{NN} = \Omega^{+}_{NN+}\,\Omega_{NN-} = \tilde{\Omega}^{+}_{NN+}\,\tilde{\Omega}_{NN-} \tag{20}$$

and the unitary operator $A(NN,\pi)$,

$$A(NN,\pi) := \tilde{\Omega}_{NN\pm}\,\Omega^{+}_{NN\pm} \tag{21}$$

transforms $M_{NN,\pi}$ into $\tilde{M}_{NN,\pi}$,

$$\tilde{M}_{NN,\pi} = A(NN,\pi)\cdot M_{NN,\pi}\cdot A^{-1}(NN,\pi) \tag{22}$$

For the cluster consisting of nucleon a and the pion we define

$$\vec{P}_{a\pi} := \vec{p}_a + \vec{p}_\pi \tag{23}$$

$$k_\pi := L(\vec{P}_{a\pi}/M^0_{a\pi})\,p_\pi \tag{24}$$

State vectors in $\mathcal{H}_N \oplus \mathcal{H}_\Delta \oplus \mathcal{H}_N \otimes \mathcal{H}_\pi$ are represented by 3-component functions $\psi_N(\vec{p}_a)$, $\psi_\Delta(\vec{p}_\Delta)$, $\psi_{N\pi}(\vec{P}_{a\pi},\vec{k}_\pi)$. The operator M_a is given by the block matrix

$$\hat{M}_{a\pi} = \begin{bmatrix} m_N & 0 & 0 \\ 0 & m_\Delta & v_\Delta(\vec{k}_\pi) \\ 0 & v^*_\Delta(\vec{k}_\pi) & W(\vec{k}_\pi)\cdot\delta(\vec{k}'_\pi-\vec{k}_\pi)\cdot(\vec{k}'_\pi|\,v_{N\pi}\,|\vec{k}_\pi) \end{bmatrix} \tag{25}$$

where

$$W(\vec{k}) = (\vec{k}^2+m^2)^{1/2} + (\vec{k}^2+m_N^2)^{1/2} \tag{26}$$

If we add a spectator nucleon b the Hilbert space is

$$\mathcal{H} = (\mathcal{H}_N \oplus \mathcal{H}_\Delta \oplus \mathcal{H}_N \otimes \mathcal{H}_\pi) \otimes \mathcal{H}_N \tag{27}$$

$G_{a\pi}$ is defined in the obvious manner and

$$G_{a\pi,b} = G_{a\pi} + G_b \tag{28}$$

It follows that

$$\Omega_{a\pi,b\pm} = \Omega_{a\pi\pm} \tag{29}$$

As an operator on functions $\vec{k}_\pi$, $\vec{P}$ and $\vec{q}_b$, defined by

$$q_b = L(\vec{P}/M_0)\, p_b \tag{30}$$

$\vec{X}_o$ is again $i\nabla_p$. The mass operator $\tilde{M}_{a\pi,b}$ defined by

$$\tilde{M}_{a\pi,b} := (\vec{q}_b^{\,2} + \tilde{M}_{a\pi}^2)^{1/2} + (\vec{q}_b^{\,2} + m_N^2)^{1/2} \tag{31}$$

commutes with $\vec{X}_o$ if

$$(\vec{q}'_\pi \vec{P}' | \tilde{M}_{a\pi} | \vec{P}\, \vec{q}_\pi) := \delta(\vec{P}'-\vec{P}) \cdot \delta(\vec{q}'_\pi - \vec{q}_\pi)\, \hat{M}_{a\pi} \tag{32}$$

where $\hat{M}_{a\pi}$ is the block matrix (25). The representations $\tilde{G}_{a\pi,b}$ and $G_{a\pi,b}$ are scattering equivalent.

The complete mass operator $\tilde{M}$ is then

$$\tilde{M} = \tilde{M}_{NN,\pi} + \tilde{M}_{a\pi,b} + \tilde{M}_{b\pi,a} - 2M_0 + V_0 + V'' \tag{33}$$

where V_0 is a two-body interaction in $\mathcal{H}_{NN} \oplus \mathcal{H}_{N\Delta}$ and vanishes in $\mathcal{H}_{NN\pi}$, and V'' is a three-body interaction in $\mathcal{H}_{NN\pi}$ with transition matrix elements to $\mathcal{H}_{NN}$. Betz and Lee[2] have fitted the parameters of a model of this type to pion-nucleon scattering and to both elastic and inelastic nucleon-nucleon scattering. The application to pion-deuteron scattering produced reasonable results.

We now come to the problem of electromagnetic interactions. What can be done to combine the quantum electrodynamics of photons and electrons with a direct-interaction hadron model? Is it possible to add to the Hamiltonian the standard interaction of the form

$$H' = \int d^3x \cdot j_h^\mu(\vec{x}) \cdot A_\mu(\vec{x}) \tag{34}$$

where $j_h^\mu(\vec{x})$ is a hadron current density? The following lemma should be useful.

Lemma: Assume that $\vec{J},\vec{P},H,\vec{K}$ satisfy the Poincaré commutation relations and define

$$H' := \int d^3x \cdot \eta(\vec{x}) \tag{35}$$

$$\vec{K}' := \int d^3x \cdot \vec{x} \cdot \eta(\vec{x}) \tag{36}$$

where $\eta(0)$ commutes with $\vec{J}$ and $\vec{K}$,

$$[\vec{J}, \eta(0)] = [\vec{K}, \eta(0)] = 0 \tag{37}$$

and

$$\eta(\vec{x}) = e^{-i\vec{P}\cdot\vec{x}} \cdot \eta(0) \cdot e^{i\vec{P}\cdot\vec{x}} \tag{38}$$

Then the generators $\vec{J}, \vec{P}, H+H', \vec{K}+\vec{K}'$ satisfy the commutation relations (I.3)-(I.9) provided $\vec{K}'$ commutes with H' and the components of $\vec{K}'$ commute with each other,

$$[\vec{K}', H'] = \frac{1}{2} \int d^3x \int d^3x' \, (\vec{x}-\vec{x}') \cdot [\eta(\vec{x}), \eta(\vec{x}')] = 0 \tag{39}$$

$$[K_i', K_j'] = \frac{1}{2} \int d^3x \int d^3x' \sum_\kappa \epsilon_{ij\kappa} (\vec{x}-\vec{x}')_\kappa [\eta(\vec{x}), \eta(\vec{x}')] = 0 \tag{40}$$

From (35), (37) and (38) it follows that

$$[\vec{P}, H'] = [\vec{J}, H'] = 0 \tag{41}$$

and from (35), (36), (37) and (38) we have

$$[J_i, K_j'] = i \sum_k \epsilon_{ijk} K_k' \tag{42}$$

and

$$[K_i', P_k] = i \delta_{ik} H' \tag{43}$$

From (38) and (I.9) it follows that

$$[K_i, \eta(\vec{x})] = x_i [H, \eta(\vec{x})] \tag{44}$$

and hence

$$[K_i, H'] + [K_i', H] = 0 \tag{45}$$

Thus Eq. (39) is necessary and sufficient for

$$[\vec{K}+\vec{K}', H+H'] = i\vec{P} \tag{46}$$

to hold. From (44) it follows that

$$[K_i, K_k'] + [K_i', K_k] = 0 \tag{47}$$

Thus Eq. (40) is necessary and sufficient for

$$[K_i + K_i', K_k + K_k'] = i \sum_m \epsilon_{ikm} J_m \tag{48}$$

to hold. Obviously local commutativity, $[\eta(\vec{x}), \eta(\vec{x}')] = 0$, is sufficient for (39) and (40).

For practical purposes the conditions (39) and (40) can be ignored for the large number of applications where H' is a perturbation and the first order is sufficient. An example is high-energy electron-nucleus scattering in the one-photon exchange approximation.

Let G_{em} and G_h be respectively the Poincaré generators of quantum electrodynamics (electrons, positrons and photons) and for a system of hadrons with direct interactions. Then the operator $\eta(\vec{x})$,

$$\eta(\vec{x}) = j^\mu(\vec{x}) \cdot A_\mu(\vec{x}) \tag{49}$$

satisfies (37) if $j^\mu(\vec{x})$ is a hadron current density satisfying

$$U_h(\Lambda) \cdot j^\mu(0) \cdot U_h^{-1}(\Lambda) = \Lambda^\mu{}_\nu \, j^\nu(0) \tag{50}$$

and $A_\mu(\vec{x})$ is the Maxwell field,

$$U_{em}(\Lambda) \cdot A^\mu(0) \cdot U_{em}^{-1}(\Lambda) = \Lambda^\mu{}_\nu \cdot A^\nu(0) \tag{51}$$

No general prescription is known for the construction of the current density for directly interacting hadrons. Approximate solutions can be attained by formal expansion in inverse powers of the velocity of light[3,4]. Classical theory suggests that the construction of a covariant conserved current may be related to particle position operators satisfying the world-line conditions. Canonical coordinates cannot satisfy the world-line conditions exactly[5] but they can be satisfied approximately[6] in a formal expansion in inverse powers of the velocity of light to order $1/c^2$. The approximate construction of covariant conserved currents and the approximate world-line conditions are indeed closely related in that approximation but the approximations do not seem to point to an exact relation.

A word of caution is in order concerning expansions in powers of

$1/c^2$. The velocity of light is a convenient tage, but its power does not by itself measure the size of terms in the expansion. The relevant physical quantities are the velocities of the particles. In a classical theory the expansion is justified if the velocities of all particles are small compared to the velocity of light everywhere on each orbit. In a quantum mechanical theory the expansion is in powers of the unbounded operator $\vec{p}^2/(mc)^2$ $(c=1)$. An expansion of $(\vec{p}^2+m^2)^{1/2}$ in powers of $\vec{p}^2/m^2$ must be justified by restrictions on acceptable states ψ. The error of a nonrelativistic approximation

$$\| \left((m^2+\vec{p}^2)^{1/2} - m - \frac{\vec{p}^2}{2m} \right) \psi \| \qquad (52)$$

may be acceptably small. The errors of successive improvements

$$\| \left((m^2+\vec{p}^2)^{1/2} - m - \frac{\vec{p}^2}{2m} + \frac{1}{8}\frac{\vec{p}^4}{m^3} \right) \psi \|$$

and

$$\| \left((m^2+\vec{p}^2)^{1/2} - m - \frac{\vec{p}^2}{2m} + \frac{1}{8}\frac{\vec{p}^4}{m^3} - \frac{\vec{p}^6}{16m^5} \right) \psi \|$$

may or may not be successively smaller. Momentum-space wavefunctions typically decrease as some power of the momentum for large p . Depending on the nonrelativistic approximation may be quite adequate, but the improved versions are much worse. Or perhaps the first relativistic correction is still an improvement. The moral of this story is simple: Quit while your are ahead! Don 't press your luck! Also it may be legitimate to expand in powers of some momenta and not others. In the applications of the $NN\pi$ model discussed earlier the pion velocities are usually relativistic, baryon velocities are usually but not always nonrelativistic.

Expansion in powers of $1/c^2$ have been widely used for the purpose of constructing compatible interaction terms for $\vec{K}$ and H without recourse to the Bakamjian-Thomas construction. The procedure has yielded satisfactory results to order $1/c^2$. In that approximation cluster separability, the world line conditions and a reasonable relation to conventional field theories are all closely related.

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THE MULTITIME COVARIANT FORMALISM OF RELATIVISTIC DYNAMICS

Ph. Droz-Vincent
College de France
Chaire de Physique Mathématique
11, Place Marcelin Berthelot
75231 PARIS Cedex 05

1. N-Body Relativistic Systems

In Predictive Mechanics, the basic equations of motion form a true differential system [1] .

$$\frac{dx_a^\mu}{d\tau_a} = v_a^\mu \quad , \qquad \frac{dv_a^\mu}{d\tau_a} = \mathfrak{Z}_a^\mu (x_1, \ldots, x_N, v_1, \ldots, v_N) \tag{1.1}$$

where the generalized accelerations $\mathfrak{Z}$ are submitted to the Predictivity condition.

$$\left(v_a \cdot \partial_a + \mathfrak{Z}_a \cdot \frac{\partial}{\partial v_a} \right) \mathfrak{Z}_b^\alpha = 0 \quad , \quad a \neq b \tag{1.2}$$

Phase space is the bundle $(T(M^4))^N$ equipped with the natural coordinates $x_1, \ldots x_N ; v_1, \ldots v_N$. Whenever no confusion is possible we drop the greek indices $\alpha = 0,1,2,3$ signature $+ ---$

$$a,b = 1, \ldots , N .$$

No summation over repeated particle indices, except if explicitly specified. Condition (1.2) is stronger than the simple Frobenius integrability condition, since it insures individuality: the solutions have the form

$$x_a = x_a (\tau_a) \tag{1.3}$$

which allows for world-lines. In the first presentation of this formalism, we assumed additionally

$$\mathfrak{Z}_a \cdot v_a = 0 \tag{1.4}$$

which implies that each v_a^2 is constant in the motion. This constant is then identified with the squared mass m_a^2 and each τ_a is proportional to the proper time, viz.

$$\tau_a = s_a / m_a \tag{1.5}$$

The framework can be generalized provided the constancy of masses is recovered somewhere, which is the case for hamiltonian systems. Accordingly eq. (1.4) can be regarded as subsidiary. When (1.4) is dropped the parameters τ_a are no longer affine parameters: (1.5) is no longer valid [2].

The dynamical system described by eq. (1.1) is equivalent to a (local) N-parameter abelian group g_N acting in $(T(M_4))^N$. This group of multi-time translations has the infinitesimal generators

$$X_a = v_a \cdot \partial_a + 3_a \frac{\partial}{\partial v_a} \tag{1.6}$$

(Geometrically each X_a defines a vector field).
The predictivity condition (1.2) simply reads

$$[X_a, X_b] = 0 \tag{1.7}$$

and the orbits of g_N are N dimensional surfaces, they provide a foliation of phase space. The projection of each orbit on $(M_4)^N$ yields the cartesian product of N world-lines (i.e. a world-surface). When (1.4) holds, fixing the positive value of m_a^2 selects a 7N dimensional submanifold which is invariant by g_N.
Suppose we are given a multitime dynamical system, satisfying both (1.2) and (1.4). In order to have a hamiltonian formalism we should look for a symplectic form invariant by g_N.
Since a famous theorem [3] forbids to require that the positions $x_1, \ldots, x_N$ be canonical variables, the matter is ambiguous and additional prescriptions are needed for the hamiltonization. It happens that the inverse procedure is more easy to carry out. Thus constructive and practical motivations lead to consider a priori a hamiltonian system in an abstract phase space where a set of canonical coordinates are $q_1, \ldots, q_N, p_1, \ldots, p_N$.
Multitime Hamilton equations of motion can be written, but they are eventually identified with eq. (1.1) and finally a dynamical system is recovered in terms of the natural coordinates $x_1, \ldots, x_N, v_1, \ldots, v_N$.
The key of this identification is the transformation $q,p, \rightleftharpoons x,v$

from canonical to natural coordinates. Of course, the procedure is not mathematically unique, and physically reasonable prescriptions are invoked in order to select a dynamical system.

In this a priori hamiltonian approach[4] one starts from N covariant hamiltonians $H_1 \ldots H_N$ which are functionally independent and strongly commute among themselves

$$\{H_a, H_b\} = 0 \tag{1.8}$$

Example: The free hamiltonians are $\overline{H}_a = \frac{1}{2} p_a^2$. When interaction is present we have $H_a = \overline{H}_a + V_a$ where V_a are pseudo-potential terms chosen as to satisfy condition (1.8). Naturally q_a transform as points in Minkowski space M_4 , whereas the p_a transform as four vectors. Standard Poisson brackets are assumed

$$\{q_a^\alpha, p_{b\beta}\} = \delta_{ab} \cdot \delta^\alpha_\beta$$

and the Poincaré algebra is generated by $P = p_1 + \ldots p_N$ and $M = q_1 \wedge p_1 + \ldots q_N \wedge p_N$. The hamiltonians are not directly related with the energy but rather with the masses. They generate the Liouville operators (or equivalently vector fields) X_a through the definition

$$X_a f \equiv \{f, H_a\} \qquad \forall f. \tag{1.9}$$

From (1.9) it is obvious that these Liouville operators satisfy (1.7). Thus they generate an abelian group. The orbits of this group are just the N dimensional integral surfaces defined by the Hamilton-like equations of motion

$$\frac{\partial q_a}{\partial \tau_b} = \{q_a, H_b\} \ , \qquad \frac{\partial p_a}{\partial \tau_b} = \{p_a, H_b\} \tag{1.10}$$

Note that, in general, solving eq. (10) yields each q_a as a function of all the parameters τ_b [5] .

No world-lines have appeared so far. But now, if we find the quantities $x_1, \ldots, x_N$ (non degenerate and transforming like the $q_1, \ldots, q_N$) satisfying

$$\{H_a, x_b^\alpha\} = 0 \ , \qquad a \neq b \tag{1.11}$$

then, an appropriate change of variables permit to identify X_a of (19.) with the X_a of eq. (1.6), and eq. (1.10) is finally equiva-

lent to (1.1). Since (1.7) is satisfied by construction, the solutions certainly have the form (1.3) (world lines).

Hint of the proof:

Define

$$v_a = \{x_a, H_a\} \tag{1.12}$$

$$\mathfrak{z}_a = \{v_a, H_a\} \tag{1.13}$$

and compute $\frac{\partial v_a}{\partial \tau_b}$ from (1.10) taking (1.11) into account.

Important remarks

a) Practically H_a are given functions of the canonical variables and the position equations (1.11) have to be solved with respect to the unknown functions $x_a(q_1,..,q_N, p_1,..,p_N)$.

b) Whereas (1.7) is satisfied by construction, in contrast eq. (1.4) is generally not valid for $\mathfrak{z}$ and v obtained from (1.12) (1.13).

c) Eq. (1.11) admit infinitely many solutions. Playing with this arbitrariness permits, in principle, that we choose the positions x with enough care in order to satisfy (1.4), if we really wish to do it. Nevertheless, for a pragmatic reason of simplicity, we prefer to drop the condition (1.4) .

This enlargement of the formalism allows for evolution parameters which are generally distinct from the proper times. But we gain simplicity in the construction of models and especially in the solving of eq. (1.11) .

Fortunately the hamiltonians H_a provide N constants of the motion. It remains possible to identify their numerical values with $\frac{1}{2} m_a^2$ and this fixes the reparametrization of the world-lines:

$$\frac{d\tau_a}{d\sigma_a} = \sqrt{\frac{2 H_a}{v_a^2}} \tag{1.14}$$

where $\sigma_a = s_a / m_a$

Naturally, we accept only the solutions in which v_a^2 never vanish.

d) We insist: in the a priori hamiltonian approach the model has no physical meaning unless a solution of (1.11) is specified.

A unique solution to the position equations can be selected by requiring that all the $x_a - q_a$ vanish on some suitable Cauchy surface (Σ) of dimension $7N+1$. Of course (Σ) must not be characteristic (never invariant under the transformations generated by any set of $N-1$ vector fields taken among the X_a) .

Besides, (Σ) shall be invariant under the Poincaré group and particle permutations (this insures that solving (1.11) preserves these symmetries).
For instance, we have suggested [6] to define (Σ) by

$$P \cdot (q_a - q_b) = 0 \qquad (1.15)$$

Some arguments from Mutze theorem [7] and constraint relativistic dynamics are against the use of center of mass variables, in view of cluster separability. Hence, for $N > 2$, the above choice is subject to controversy and might be replaced if necessary. Anyway, previous to the choice of (Σ), the general N-body case faces the algebraic difficulty of constructing explicitly admissible interactions which satisfy the commutativity condition (1.8) [8]. As pointed out by F. Rohrlich [9], H. Sazdjian [10] and I.T. Todorov [11], the very important requirement of separability makes this matter more complicated. Moreover, discussing the asymptotic behavior of the potentials in terms of the canonical variables $q_a - q_b$ can be misleading in so far as the exact relationship between the q and the positions is not exhibited.
However, substantial progress have been made in the constraint formalism. Their possible adaptation to the present formalism could deserve some interest. But the questions specific of the general case $N > 2$ will not be discussed in details here.

From now on, we shall consider the simple case of two-body systems.

2. Two Body Systems

Each motion of the system is represented by a two-dimensional orbit in phase space. Its projection onto $M_4 \times M_4$ is a world-surface i.e. the cartesian product of two world-lines. This world-surface is the intrinsic history of the system. But any observer will slice space-time by a sequence of parallel hyperplanes. The slicing of the world-lines implies that this observer picks up, from the world-surface, a one-parameter sequence of couples x_1, x_2. This sequence of x_1, x_2 is the equal-time history associated with this observer. Its lift in phase space is a curve drawn on the orbit.

We could consider an arbitrary observer, independent with respect to the system. This point of view would introduce a constant time-like direction U^α and the equal-time surface of the observer, by the equation

$$U \cdot (x_1 - x_2) = 0$$

In principle it is possible to construct the single-parameter description associated with any such observer, and this should be explicitly carried out in order to make contact with the work of other authors.

We have preferred to give the single-parameter description associated with an observer attached to the center-of-mass [2]. In this equal-time description the slicing of space-time selects couples x_1, x_2 satisfying

$$P \cdot (x_1 - x_2) = 0 \tag{2.1}$$

Since we have required that x_1, x_2 and q_1, q_2 respectively coincide on the surface

$$(\Sigma) \qquad P \cdot (q_1 - q_2) = 0 \tag{2.2}$$

this Cauchy surface turns out to be also the equal-time surface. Let us define $z = q_1 - q_2$, $r = x_1 - x_2$. By integrating the equations of motion we obtain $P \cdot z$ in terms of the evolution parameters τ_1, τ_2.

The points of (Σ) satisfy a relation of the form

$$P \cdot z(\tau_1, \tau_2) = 0 \tag{2.3}$$

Defining $\lambda = \tau_1 + \tau_2$ we put eq. (2.3) in the equivalent form

$$\tau_1 = f_1(\lambda) \quad , \qquad \tau_2 = f_2(\lambda) \tag{2.4}$$

where $f_1 + f_2 \equiv \lambda$, which preserves the democracy of particles. Then, provided we solve the equations of motion (which yields the evolution of q_1 and q_2) a parametric representation of the world-lines (corresponding to the equal-time description) is given by

$$\begin{aligned} x_1 &= q_1 (f_1(\lambda), f_2(\lambda)) \\ x_2 &= q_2 (f_1(\lambda), f_2(\lambda)) \end{aligned} \tag{2.5}$$

without solving the position equations [12].

Though neither X_1, nor X_2 leaves (Σ) invariant, a suitable combination Y of them does: $Y(P \cdot z)$ vanishes. For instance in the case of a central-like potential we have

$$Y = \frac{1}{P\cdot p_1} X_1 + \frac{1}{P\cdot p_2} X_2 \qquad (2.6)$$

Note that the coefficients of this combination have obviously something to do with the fixations of constraint relativistic dynamics. The vector field Y is tangent to the lift of the equal-time history. In the above example

$$f_1 = \frac{P\cdot p_2}{P^2}\lambda + f_0$$

$$f_2 = \frac{P\cdot p_1}{P^2}\lambda - f_0 \qquad f_0 = \text{const.}$$

As proved in details by several authors [13] the singular lagrangian approach [14] can be incorporated into this framework. The relation with constraint dynamics has been analyzed and clarified by L. Lusanna [15].

3. Solving Position Equations: Cauchy Surface Versus Asymptotic Conditions

We decided [2] to select the positions obtained from eq. (1.11) by the vanishing of $x_1 - q_1$ and $x_2 - q_2$ on (Σ). This boundary condition is natural and leads to abundant simplifications. It permits the contact with non-relativistic mechanics through a description in the center-of-mass frame and seems to be the best choice insofar as confinement is concerned. (This was our initial motivation).

Alternatively, an asymptotic condition looks reasonable as well in a different context. As soon as it was recognized that the positions cannot be canonical, R.N. Hill and E.H. Kerner suggested to fix their relationship with canonical variables by the requirement that they become asymptotically canonical, when the spatial separation between particles goes to infinity [16]. In the same spirit, the hamiltonization procedure used by L. Bel and his co-workers [17] rests on asymptotic conditions.

When scattering particles are considered, then asymptotic conditions seem to be more appropriate than the boundary condition on a Cauchy surface.

Since distinct solutions of (1.11) lead to inequivalent dynamics, it may be questioned whether the system obtained from equal-time conditions on (Σ) has an admissible asymptotic behavior. In particular, P^ν and $M_{\mu\nu}$ should coincide with the free-particle form for infinite spatial separations [18]. Reminding that the true natural (not

canonical!) momenta are not v_1, v_2 but rather

$$w_1 = \frac{d\tau_1}{d\sigma_1} v_1 \quad , \qquad w_2 = \frac{d\tau_2}{d\sigma_2} \cdot v_2$$

with $d\tau_a/d\sigma_a$ given by (1.14) it is crucial to check if: For any "good behaving" potentials

$$w_1 + w_2 \longrightarrow P$$
$$x_1 \wedge w_1 + x_2 \wedge w_2 \longrightarrow M \qquad (3.1)$$

when

$$|\vec{r}^2| \longrightarrow \infty$$

Suitable asymptotic conditions would have implicitly incorporated (3.1).

In contrast, as we start from Cauchy conditions on (Σ), there is absolutely no evidence that (3.1) is satisfied.

In fact, both choices have their limitations: asymptotic conditions do not provide easily existence theorems. Moreover, they require that the interaction is fastly enough decreasing at (spatial) infinity and do not apply to the harmonic oscillator. Therefore we do not abandon the principle of equal-time conditions, specified on (Σ). But we face the problem of checking that the resulting dynamics is not pathologic at infinity.

It is not proved but perhaps reasonable to expect that for a suitable class of potentials, the dynamics obtained from

$$(x_1 - q_1)_{|\Sigma} = (x_2 - q_2)_{|\Sigma} = 0 \qquad (3.2)$$

does satisfy (3.1).

The best way to conclude would be explicitly solving the position equations (1.11). We present recent investigations about this problem. For the present time, we shall not discuss the validity of conjecture (3.1) but simply whether it is true that

$$(x_1 - q_1) \longrightarrow 0 \quad , \quad (x_2 - q_2) \longrightarrow 0 \qquad (3.3)$$

for $\quad |\vec{r}^2| \longrightarrow \infty$

Let us consider an unipotential two-body system [19] ($V_1 = V_2 = V$) and assume a central-like interaction of the form

$$V = F(\tilde{z}^2) \tag{3.4}$$

Since the relative canonical variables $\tilde{z}, \tilde{y}$ remain in a constant plane, we try a solution of the form

$$x_1 = q_1 + \frac{p_{p2}}{p^2}(-\phi_2 \tilde{z} - \psi_2 \tilde{y}) \tag{3.5}$$

$$x_2 = q_2 + \frac{p_{p1}}{p^2}(\phi_1 \tilde{z} + \psi_1 \tilde{y}) \tag{3.6}$$

This method was used first in the harmonic case, but now we do not restrict the dependence of ϕ_a, ψ_a in their arguments. The position equations become a partial differential system in the unknown functions ϕ_a, ψ_a.
Practically it is enough to find, for instance, ϕ_1, ψ_1. Particle exchange in the formulae will provide the expressions for ϕ_2, ψ_2. The equations for ϕ_1, ψ_1 involve only the Liouville operator X_1. We set

$$\theta_1 = \frac{p_z}{p_{p1}} \quad , \qquad \theta_2 = -\frac{p_z}{p_{p2}}$$

hence we compute

$$X_1 \theta_1 = 1 \quad , \qquad X_2 \theta_2 = 1 \tag{3.7}$$

Let us provisionally drop the indice 1 in X, ϕ, ψ, θ and determine x_2 according to (3.6).

Apply X to (3.6), develop the result on the linearly independent vectors $\tilde{z}, \tilde{y}$. Defining F^* as $dF/d(\tilde{z}^2)$ we obtain the system

$$\phi = -X\psi \tag{3.8}$$

$$(X^2 + 2F^*)\psi + 2F^*\theta = 0 \tag{3.9}$$

with the initial condition: ϕ and ψ vanish for $\theta = 0$. Note that $\theta = 0$ is nothing but the equation which defines (Σ). In (3.8) (3.9) F^* depends only on $\tilde{z}^2$. In order to render the system more explicit, we must compute $X\phi$ and $X\psi$ in terms of the derivatives of ϕ and ψ.

Fortunately θ satisfies (3.7) and $X\tilde{z}^2$ depends only on $\tilde{z}^2$ and two constants of the motion, viz.

$$\ell^2 = \tilde{z}^2\tilde{y}^2 - (\tilde{z}\tilde{y})^2 \tag{3.10}$$

which is non-negative since $\tilde{z}$ and $\tilde{y}$ are spacelike, and

$$N = \tilde{y}^2 + 2V \tag{3.11}$$

which is certainly negative if V vanishes anywhere (including ∞). Indeed we have

$$X\,\tilde{z}^2 = -2\,\tilde{z}\cdot\tilde{y} \tag{3.12}$$

and $\tilde{z}\cdot\tilde{y}$ can be expressed in terms of $\tilde{z}^2$, ℓ^2, N by elimination of $\tilde{y}^2$. (Doing this we consider ℓ^2 and N as phase space functions. They are not given numerical values).

Let us introduce the variable

$$\zeta = -\tilde{z}^2$$

From (3.12) we derive

$$X\zeta = -2\varepsilon\sqrt{\zeta(2V-N) - \ell^2} \tag{3.13}$$

with ε = sign of $\tilde{z}\cdot\tilde{y}$.

From now on we may require that ϕ and ψ depend only on

$$\theta, \zeta, \ell^2, N$$

Since ℓ^2 and N are constants of the motion, only $X\theta$ and $X\zeta$ give a contribution to $X\phi$ and $X\psi$. We have $F^* = -\,dF/d\zeta$. From (3.4) and (3.13) we see that only θ, ζ, N, ℓ^2 appear explicitly in (3.8) (3.9). Variables other than these ones can be ignored, and ℓ^2 and N behave practically like constants in this problem. The system (3.8) (3.9) takes immediately the normal form in θ, thus a certain solution exists which vanishes for $\theta = 0$. By unicity we know that it is the one we look for.

Now the point is whether, for a certain class of F, this local solution can be extended to arbitrarily large values of ζ and vanishes again for $\zeta \to \infty$. <u>If this is true</u>, symmetry under particle exchange provides a similar behavior of ϕ_2, ψ_2. Then the positions determined by (1.11) satisfy

$$x_1 - q_1 \longrightarrow 0\,, \qquad x_2 - q_2 \longrightarrow 0$$

for $\zeta \to \infty$. As a result $\tilde{r}^2 \sim \tilde{z}^2$ and, in phase space, the asymptotic region $\zeta = \infty$ can be identified with (at least a part of) spatial infinity ($\tilde{r}^2 = \infty$) . Now the vanishing of the potential for $\zeta \to \infty$ could be interpreted a posteriori as a true separability property and (3.3) would be satisfied. Finally in order to check if the above situation may really occur, we should investigate the asymptotic behavior (in ζ) of the solution which vanishes at $\theta = 0$, for the system (3.8) (3.9). This task has not been achieved yet. Only a formal expansion in powers of θ has been obtained so far [20] . Its coefficients vanish for $\zeta \to \infty$, provided F is analytic and vanishing at $\zeta = \infty$.

Alternative methods exist for solving position equation. As observed by Iranzo, Llosa, Marques, Molina, these equations can be solved whenever the canonical equations of motion can be explicitly integrated [21] . Applying their argument to the case of a potential which vanish for $\tilde{z}^2 > R = \text{const.}$ (F is not analytic, but has compact support) we see easily that $x_1 = q_1$ and $x_2 = q_2$ are the solutions in a region $\tilde{z}^2 > A_{(R)}$, $p\tilde{z} < a_{(R)}$ (x_1 and x_2 are unknown but different in the region where interaction takes place). Recently L. Lusanna has given a tractable exponential formula [22] .

Let us point out that this formula applies with θ_1 and θ_2 as we defined, not only in the harmonic case, but for all central potentials: the quantities $X_a \theta_b$ are the same.

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SECOND QUANTIZATION OF DIRECTLY INTERACTING PARTICLES

Ph. Droz-Vincent

College de France
Chaire de Physique Mathématique
11, Place Marcelin Berthelot
75231 PARIS Cedex 05

Introduction

We undertake to formulate the second quantization of relativistic dynamics along the lines of action-at-a-distance theory.

It can be objected that the relativistic theory of direct interactions is not yet sufficiently mature to permit such an attempt.

But the question of particle creation is often raised a priori against the idea of N-body relativistic dynamics.

For this reason, at least, it is essential to investigate whether N-body dynamics can be naturally continued by a more general theory which accounts for particle creation (or anihilation).

We are aware of the mathematical difficulties that this program may involve in its developments. But it is already important to see how a consistent picture is at least conceptually possible.

Crudely speaking the way we suggest consist in quantum field theory without fields and, in general, without locality. We stress the fact that second quantization can be conceived independently from the (technical) concept of local field operators, in a Fock space scheme which incorporates N-body dynamics (with arbitrary N) as an intermediate step.

This view departs manifestly from the conventional habits of Q.F.T. But it is in agreement with the old ideas of Heisenberg [1] about the possibility of a description directly in terms of the scattering operator without explicit mention of the field.

In Section 1. General properties of N-body dynamics are recalled.

In Section 2. Second quantization is considered in a scheme where the number of particle is still conserved.

Although not yet realistic, the situation described therein is of a certain pedagogical interest, especially it countains the case of free

particles.

In Section 3. We display some principles for breaking the particle number conservation. At this stage we have constructed no satisfactory example yet. We have just sketched the main lines of a forthcoming theory which seems to emerge naturally, irrespectively of the technical difficulties it may involve.

1. N-Body Quantum Mechanics

Relativistic quantum mechanics is based on covariant wave equations. In particular, in quantum predictive mechanics these wave equations look formally like eigenvalue equations. The hamiltonians are operators and their eigenvalues are supposed to be $\frac{1}{2}$ of the squared masses.

This simple principle permits to recover a system of N independent K-Gordon equations in the case of free particles. Coupling terms appear when interaction is present [2].

But a rigorous statement of this principle requires some care.

Indeed, even for free particles the wave function cannot be in the Hilbert space $L^2(\mathbb{R}^{4N})$, whereas the standard theory of operators and eigenvalues is generally understood as taking place within some Hilbert space.

Naturally, in the free case at least, the wave function may belongs to a Hilbert space: The solutions corresponding to given values of the masses have a well-known scalar product defined through Fourier transform on the N-uple hyperboloid which correspond to these masses. This structure is related with the usual probabilistic interpretation [3]. But such a Hilbert space is of no use for our purpose: it depends on the masses and the hamiltonian operators act completely trivially on it: they reduce to the identity multiplied by the corresponding mass.

Therefore, in so far as one is concerned with having just a mathematical framework describing operators, eigenvectors, eigenvalues and all that, the natural Hilbert space is $L^2(\mathbb{R}^{4N})$. Fortunately the conventional technique of Hilbert spaces has been suitably enlarged and completed by the introduction of rigged Hilbert spaces. This concept introduced by Gelfand allows to circumvent the above difficulty [4]: Though the wave function cannot be in L^2, it is generally a tempered distribution . Thus the largest space in which any function we consider is supposed to be is well-defined and we deal with the triplet

$$\mathcal{S} \subset L^2 \subset \mathcal{S}^*(\mathbb{R}^{4N})$$

A "time" dependent formalism is possible also. In that case, the wave equations of the form Klein-Gordon + coupling are replaced by a relativistic Schrödinger system involving N evolution parameters.

This time dependent formalism is more general and permits to have a wave function in L^2 if on wishes. We have used it recently to give (provisionally in L^2) an axiomatics of scattering [5].

But, in contradistinction with the point of view of Horwitz and Rohrlich [6], our interpretation is that wave-functions in L^2 are off the mass-shell anyway. We do not exclude them completely. Rather we consider that these idealized objects are conceptually useful, but we should manage that the true (observable) physical processes involves only on-shell states.

Taking seriously the wave-equations (Klein-Gordon + coupling) as eigenvalue equations, we are obliged to consider the space of solutions not as isolated, but as imbedded into the larger space of tempered distributions $\mathcal{S}'$. Thus the operators we consider may act in $\mathcal{S}'$ and not only in the space of solutions.

In other words, to be consistent with the idea that physical states are eigenstates (i.e. on-shell states) we are lead to construct an off-shell framework.

A system of N particles is defined by N commuting hamiltonians $H_1, \ldots\ldots, H_N$.

This principle comes out directly by quantization of predictive relativistic mechanics in its many-time formalism. Commutation is intimately related with predictivity conditions and the fact that the translations in the "time" parameters form an abelian group [7] .

It plays an essential role in the calculations involving the evolution operator, in particular in scattering theory.

In order to have a consistent theory of eigenstates it is technically essential that the hamiltonians we consider map $\mathcal{S}$ into itself [8].

Any operator A which maps $\mathcal{S}$ into itself will be said hermitian in the rigged Hilbert space $\mathcal{S} \subset L^2 \subset \mathcal{S}'$ when

$$\langle \varphi, A\psi \rangle = \langle A\varphi, \psi \rangle \qquad (1.1)$$

holds $\forall\, \varphi$ and $\psi \in \mathcal{S}$.
Then, in the wave-equations

$$H_a \psi = \tfrac{1}{2} m_a^2 \psi \qquad (1.2)$$

ψ is a generalized eigenvector of H_a for the eigenvalue $\frac{1}{2} m_a^2$.

This situation is realized in particular by the Free particle hamiltonians

$$\bar{H}_a = -\frac{1}{2}\Box_a \tag{1.3}$$

But interacting hamiltonians satisfying the above assumption do exist. Example: If B is unitary and maps $\mathcal{S}$ into itself we can take:

$$H_a = B\,\bar{H}_a\,B^{-1} \tag{1.4}$$

(Of course this simple construction does not garantee cluster separability).

In view of relationship with the time-dependent formalism we introduce N real parameters $\tau_1, \ldots, \tau_N$ and the evolution operator:

$$U_{\tau_1,\ldots\tau_N} = \exp i \sum \tau_a H_a \tag{1.5}$$

For the free hamiltonians (1.3) it is easy to check that U maps $\mathcal{S}$ into itself. This property holds obviously also when (1.4) is valid. Whenever it will be necessary we shall assume that U is unitary in the rigged Hilbert space, i.e. is unitary and maps $\mathcal{S}$ into itself. Details about the time-dependent formalism, time-dependent wave function, and the Schrödinger equation

$$-i\frac{\partial \chi}{\partial \tau_a} = H_a \chi(x_1,\ldots,x_N;\tau_1,\ldots,\tau_N) \tag{1.6}$$

have been displayed elsewhere, with a sketch of a scattering formalism in L^2 . The appearance of $\tau_1, \ldots, \tau_N$ is natural by analogy with the multi-time formalism of predictive mechanics, in the a priori hamiltonian approach. Accordingly the evolution parameters are not necessarily the proper times [9] . However they are required to be sufficiently equivalent to the proper-times in some asymptotic sense in order to allow that the asymptotic properties of the system are obtained by letting all the τ_a go to $\pm\infty$.

As this point come two remarks:

a) The control of this assumption still requires some investigations even at the classical relativistic level. This problem is provisionnally left aside.

b) As recently stressed by Rohrlich and Horwitz [6] , dealing with many parameters rises, in principle the question of the order of the limits in the definition of the wave operators. As we pointed out [5] this

problem does not appear in the single - potential (= unipotential) case. The general case is solved in Ref.[6].

The time-dependent wave function in (1.6) correspond to the function ψ of the time independent formalism through the formula:

$$\chi = \mathcal{U}\psi \tag{1.7}$$

Hence (1.6) is more general than (1.2). Indeed (1.7) implies (1.6) irrespectively of the validity of (1.2).

For brevity, the space of the solutions of (1.2) in $\mathcal{S}^*$ is called the mass-shell space and denoted by $J^N_{m_1,..,m_N}$ (resp. $K^N_{m_1,..,m_N}$) in presence of interaction (resp. for free particles)[10].

As well-known, $K^N_{m_1,..,m_N}$ is endowed with a mass depending scalar product $(\ ,\ )$ suitable for some probabilistic interpretations and not to be confused with the scalar product $\langle\ ,\ \rangle$ in $L^2(\mathbb{R}^{4N})$ [3].

The scattering formalism in L^2 has been established in complete analogy with the traditional non-relativistic axioms, except that $\tau_1,\ldots,\tau_N$ replace the absolute time of newtonian mechanics, but wave functions in L^2 are somehow unphysical, as they are off the mass shell. Hence, having in mind realistic applications, we extend the formalism to $\mathcal{S}^*$ as follows:

ASSUMPTION: Let $\Omega_\pm$ be the Moller wave operators, let S be the scattering operator. We require additionally that $\Omega_\pm$, $\Omega_\pm^*$, S, S^{-1} map $\mathcal{S}$ into itself. Then $\Omega_\pm$ and S can be continued to the whole space $\mathcal{S}^*$. For instance the extension of S is defined by :

$$\langle S\psi, \varphi\rangle = \langle \psi, S^{-1}\varphi\rangle \qquad \psi \in \mathcal{S}^*,\ \varphi \in \mathcal{S}$$

In particular $S\psi$ now makes sense when ψ is on the free mass-shell and $K^N_{m_1,..,m_N}$ is stable by action of S. This is the main point in view of applications to scattering processes [11].

Note that S is unitary in the rigged Hilbert space.

Now assume that, in addition to the above Assumption, the system has no bound state.

Consider f on the interacting mass-shell. We can write

$$f = \Omega_- g = \Omega_+ h$$

Then it can be checked that g and h are on the free mass-shell. The scalar product $(\ ,\ )$ in the free mass shell induces two scalar

products in $J^N_{m_1,..,m_N}$.

Indeed we can define:

$$(f_1, f_2)_+ = (h_1, h_2) \quad , \qquad (f_1, f_2)_- = (g_1, g_2)$$

where f_1 and $f_2 \in J^N_{m_1,..,m_N}$.

These scalar products coincide when S is unitary in the sense of (,) . We shall not discuss here the difficult problem of finding for which interacting potentials the wave-operators actually exist. Neither shall we display the perturbative procedure for computing the S operator.

Results in these domains, as well as on the subject of cluster separability are available in the constraint formalism [12] . We expect that a large part of the results which belong to the constraint approach can be adapted to predictive mechanics at the price of minor modifications.

2. Second Quantization with Constant Number of Particles

In the present work (essentially devoted to second quantization) our philosophy is to pretend that the problems relevant of N-body dynamics are solved and construct the picture which allows for creation or annihilation of particles.

According to the spirit of Action at a Distance, interaction is not described as mediated by a field. Consequently we stand out of the conventional framework of Quantum Field Theory (Q.F.T.). Intuitively, direct interactions (although able of maintaining causality) are generally not local as Q.F.T. is.

What we are doing presently may perhaps be considered as a multilocal, or non local, generalization of Q.F.T. [13].
(Yet it can be observed that the version of predictive Mechanics supported by Bel and al. provides a bridge towards local classical field theories [14] .

Anyway, comparison with Q.F.T. is relevant but difficult to carry out, since the whole formalism of Q.F.T. is based on field operators, whereas our picture is by no means founded on this concept. The suitable domain for such comparison seems to be axiomatic scattering theory, in which Q.F.T. is able of by-passing the role of the lagrangians.

Naturally, in the absence of interactions, our dynamics certainly recovers locality, and our point of view will be equivalent to that of Q.F.T. though differently formulated.

In analogy with N-body mechanics we postulate that the physical states (on the mass-shell states) are selected as eigenvectors of suitable operators.

They form a mass-shell space imbedded in a larger space which also countains off-shell states. Intuitively the states we are going to consider are superpositions of N-body states with arbitrary $N = 0, 1,2,.. \infty$. They are generally off-shell. To put this matter more precisely, for scalar particles the regular states are assumed to be terminating sequences of the form:

$$\Phi = (\varphi_0, \varphi_1, \ldots \varphi_R, 0 \ldots) \tag{2.1}$$

with $\varphi_0 \in \mathbb{C}, \ldots, \varphi_N \in \mathcal{S}(\mathbb{R}^{4N})$ and $\varphi_N = 0$ for $N > R$, R depends on Φ .

Let Γ be the space of such sequences. Its dual is the space Γ^* countaining the sequences (generally infinite) of the form

$$\Psi = (\psi_0, \psi_1, \ldots, \psi_N, \ldots.) \tag{2.2}$$

where $\psi_N \in \mathcal{S}^*(\mathbb{R}^{4N})$.

Let $\mathcal{H}$ be the hilbertian sum $\bigoplus_{N=0}^{\infty} L^2(\mathbb{R}^{4N})$ with usual conventions for $N = 0$. It turns out that Γ is a nuclear subspace of $\mathcal{H}$ and we have a rigged Hilbert space:

$$\Gamma \subset \mathcal{H} \subset \Gamma^* \tag{2.3}$$

The most general states which will occur are elements of Γ^*.

The scalar product in $L^2(\mathbb{R}^{4N})$ induces in $\mathcal{H}$ a scalar product which is also noted by $\langle , \rangle$ without risk of confusion.

This framework depends neither on the mass, nor on the model of interaction. (Assuming identical particles the masses are equal. Symmetry of the wave functions will be taken into account as lately as possible in order to avoid complications).

Note that $\mathcal{H}$ is <u>mathematically</u> analogous to the Fock space of non relativistic quantum mechanics, but we give no direct physical meaning to its scalar product.

Note also that the emergence of Γ is not quite new:
Indeed Wightman's theory starts from Γ , and Wightman functional are elements of Γ^*. [15]

For each such functional, this theory defines equivalence classes in Γ and the space of physical interest is obtained as the correspond-

ing factor space (Reconstruction theorem) [16]. Our procedure will be different.

Let us first set up a formulation in which the number of particles is preserved, although the particles undergo interaction.

Assume that we are given an infinite sequence of N-body systems, satisfying the requirements mentioned in the above section, including Poincaré invariance (which requires that the single-particle system is free). If we where to look for a realistic model this would rise the problem of constructing N-particle interactions when the binary interaction is specified. This problem is completely relevant of N-body relativistic mechanics and first quantization. It has been investigated by several authors but at the classical (rather than quantal) level principally [12].

As we warned before, in this section the typically N-body questions are provisionally considered as solved.

Accordingly, for each $N = 1,2,\ldots, \infty$, we have the operators $H^N_{a_N}$ with $a_N = 1,2,\ldots, N$.

They are hermitian in the rigged Hilbert space

$$\mathcal{S}(\mathbb{R}^{4N}) \subset L^2(\mathbb{R}^{4N}) \subset \mathcal{S}'(\mathbb{R}^{4N})$$

and commute among themselves, etc...

Having to distinguish N-particle states with different values of N, we are bound to introduce the following notations:

$\partial^N_{a\alpha} = \partial/\partial x^\alpha_a$ considered as essentially acting in the N-particle space $\mathcal{S}'(\mathbb{R}^{4N})$

$$\Box^N_a = \partial^N_a \cdot \partial^N_a \qquad \text{(no summation)} \qquad (2.4)$$

Greck indices are omitted whenever it is possible. Remind that a depends on N and the notation should exhibit this dependence to avoid confusion.

For example for a free system we have simply the operators:

$$\overline{H}^N_a = -\frac{1}{2}\Box^N_a \qquad (2.5)$$

It is natural to consider that the 2nd quantized states Ψ are <u>on the mass shell</u> when

$$\Psi = (\psi_0, \psi_1, \ldots, \psi_N, \ldots) \qquad (2.6)$$

with $\psi_N \in J_m^N$ for $N \geq 1$.

Let J_m be the space formed by these states. We reserve the notation K_m^N , K_m for the free particle case. In the free case the physical scalar product $(\ , \ )$ in each K_m^N induces a scalar product in K_m and the elements in K_m for which $(\ , \ ) < \infty$ define a Hilbert space $\mathcal{H}_m$ which provides after symmetrization and separation of the positive frequencies, nothing but the usual Hilbert space of free quantum field theory.

In the interacting case defining such a scalar product in the mass shell J_m rests on our ability to do the job in each J_m^N first . (See Section 1) .

Let us now prove that most mass shell states can be obtained as generalized eigenvectors of suitable operators.

In view of this consider $h_{a_N}^N$ defined by the formula

$$h_{a_N}^N \Psi = (0, \ldots, 0, H_{a_N}^N \psi_N, 0, \ldots) \tag{2.7}$$

for $N \geq 1$.
Let

$$a_2, a_3, \ldots, a_N, \ldots \tag{2.8}$$

be a sequence of integers with the constraint:

$$1 \leq a_N \leq N \quad , \quad \forall N$$

To this sequence we associate $M_{a_2 a_3 \ldots}$ defined as follows:

$$M\Psi = (0, -\tfrac{1}{2}\Box\psi_1 , H_{a_2}^2 \psi_2, \ldots, H_{a_N}^N \psi_N, \ldots) \tag{2.9}$$

wich means that the M are linear combinations of the h , namely:

$$M_{a_2, a_3, \ldots} = h + h_{a_2}^2 + \ldots h_{a_N}^N + \ldots \tag{2.10}$$

Owing to the assumptions made on the H_a^N , we see that the h and M , map Γ into itself, which allows to speak of eigenvectors in the rigged Hilbert space (2-3).

For each $N \neq 0$, J_m^N can be characterized as the generalized eigenspace common to all the H_a^N for the eigenvalue $\frac{m^2}{2}$.

As a result, if Ψ is of the form (2.6) with the additional condition $\psi_0 = 0$ which excludes the vacuum component, then Ψ is an eigenvector of $M_{a_2 a_3 \ldots}$ for any sequence of the form (2.8).

Therefore we are lead to define the exclusive mass-shell space J'_m as the set of Ψ in (2.6) which have no vacuum component ($\psi_0 = 0$) . It is easy to check that J'_m is the eigenspace common to all the M .

Obviously the h and the M altogether commute among themselves, each one commute with the generators of the Poincaré algebra, with the number of particles $\mathcal{N}$. Moreover, the h_a^N are transformed into one another by particle permutations, this property being assumed for the H_a^N [17] .

We interprete $M_{a_2 a_3 \ldots}$ as mass operators and $h_{a_N}^N$ as generators of the motion.

Note that even in the free case, a state on the mass-shell is not an eigenstate of the h_a^N , except if it is also a pure N-particle state for some N .

In contrast with the first quantization, the mass operators cannot be considered as generators of the motion.

This point gets clear when the evolution operator is introduced: for each N consider N real parameters $\tau_1^N, \ldots, \tau_a^N \ldots, \tau_N^N$ and the N-particle evolution operator

$U^N(\tau_1^N, \ldots, \tau_N^N)$ defined as in (1.5).

We postulate that the second-quantized evolution operator

$$U(\tau, \tau_1^2, \tau_2^2; \ldots \tau_{a_N}^N \ldots)$$

acts according to the formula

$$U\Phi = (\varphi_0, U'\varphi_1, \ldots, U^N\varphi_N, \ldots) \qquad (2.11)$$

Obviously U is labelled by the infinite sequence

$$T = \tau, \tau_1^2, \tau_2^2, \ldots \tau_{a_N}^N \ldots$$

which belongs to the additive group

$$\mathbb{R} \times \mathbb{R}^2 \times \ldots \mathbb{R}^N \times \ldots$$

and U enjoys the Abelian group property

$$U_T U_{T'} = U_{T+T'} \qquad U_0 = I \qquad (2.12)$$

It turns out that U is factorized

$$U = u^1 \cdot (u_1^2 u_2^2)(u_1^3 u_2^3 u_3^3) \ldots \tag{2.13}$$

where

$$u_a^N = \exp i \tau_a^N h_a^N \qquad \text{(No summation)} \tag{2.14}$$

Eq. (2.13), (2.14) involve no rigor problem because, like all the operators we have considered so far, they act as

$$A\Phi = (A_0\varphi_0, A_1\varphi_1, \ldots, A_N\varphi_N, \ldots)$$

where A_N acts on N-body states.

In matrix notation they are <u>block-diagonal</u>.

A complete time depending formalism is possible but will not be developed here.

Logically the scattering properties follow from the behavior of $U_T^{-1}\,\overline{U}_T$ (where $\overline{U}$ stands for the free particle evolution). But, owed to the block-diagonal form of U in (2.11) it is clear that <u>provided the wave operators $\Omega_\pm^N$ exist for each N</u> taking the $\lim_{T\to\infty}$ involves no new problem and the Möller operator of the second quantized system is just given by the formula:

$$\Omega_\pm \Phi = (\varphi_0, \varphi_1, \Omega_\pm^2\varphi_2, \ldots, \Omega_\pm^N\varphi_N, \ldots) \tag{2.15}$$

and the scattering operator S is given by a similar formula.

When Assumption I holds for each N, each K_m^N is stable by S^N, so K_m is stable by action of S [11].

When Assumption II holds $\forall N$, the scalar product $(\ ,\ )$ in K_m induces two mass depending scalar products $(\ ,\ )_\pm$ in J_m, through the mediation of in, out-states. These will coincide iff S (which is automatically unitary in $\mathcal{H}$ with $\langle\ ,\ \rangle$) is also unitary in K_m in the sense of $(\ ,\ )$.

To sum up, let us say: In so far as all the N-body interactions that we consider have a nice behavior, their second quantization yields a theory where the number of particles is preserved and this theory enjoys the same nice features.

We may note that:

i) Our operator h, M cannot emerge in the conventional approach.

ii) No use is made of field operators and we get rid of the Lagran-

gian formalism.

iii) We have replaced it by a hamiltonian formalism inherited from multitime predictive mechanics.

In particular the constants of the motion can be characterized as having vanishing commutators with the generators h_a^N. Besides, in a time depending scheme, a new state vector $|T\rangle$ depending on T can be introduced. It satisfies infinitely many Schrödinger-like equations:

$$\partial/\partial\tau_a^N \, |T\rangle = i \, h_a^N \, |T\rangle$$

The special case of free particles deserves some attention for its pedagogical value. Indeed, in this case, all the operators H_a^N, h_a^N, M, U, etc. can be explicitly constructed and the mass-shell space K_m is well-known. At this point, the contact with Q.F.T. is easy.

For instance, though field operators are by no means necessary in the above picture, they can be introduced quite well [18].

3. Second Quantization with Creation and Annihilation

The picture described in the above section still leaves the number of particles invariant. The block diagonal form of the "hamiltonian" generators h_a^N is responsible for that.

At least the advantage of this description was to emphasize the emergence of the operators h and M.

Now we are going to generalize the scheme in order to permit that creation (or annihilation) occur.

Naturally we maintain the principle that the exclusive mass-shell space J_m' is the eigenspace common to all the mass operators $M_{a_2 a_3 \ldots}$.

But we cannot assume anymore that (2.7) and (2.9) are valid. In other words, the interaction cannot be constructed by giving a sequence of N-body systems, in so far as we aim at particle creation.

Therefore we reject the H_a^N. But it is natural to make the following assumptions:

a) For each positive integer N these exist the operators $h_{a_N}^N$ and all the possible h^N commute among themselves.

We mean $[h^N, h^{N'}] = 0$ including when $N \neq N'$.

b) All the possible h^N are functionally independent.

c) They commute with the generators of the Poincaré algebra.

d) They carry Γ into itself and are hermitian in the rigged Hilbert space $\Gamma \subset \mathcal{H} \subset \Gamma^*$.

e) In view of taking into account indiscernability, we should add extra requirements which rule the behavior of the h_a^N under particle per-

mutations.

This matter will not be discussed here. Let us say simply: these extra requirements are devoted to insure that the mass operators $M_{a_2 a_3 \ldots}$ have enough symmetric eigenstates.

f) The (half squared) mass operators M are defined by eq. (2.10) and the exclusive mass-shell J_m' is the eigenspace common to all the M.

The physical states are then obtained by linear combination with the vacuum and, of course, symmetrization.

Now the generators h, and therefore the M also, are supposed to have non vanishing commutators with the particle number.

As a result, $\mathcal{N}$ is not constant in the motion, i.e. the eigenstates of M are not associated with any definite value of $\mathcal{N}$.

These assumptions lead to some comments.

First of all, we insist on commutativity in (a). This property implies that the M are constants of the motion and have common eigenvectors.

In (b) we made explicit an assumption which is usually implicit in the theory of N-body systems.

Assumption (f) raises the problem of convergence.

Defining the evolution operator by eq. (2.13), (2.14) also rises a problem of convergence.

If the infinite product (2.13) exists, then U satisfies eq. (2.12) which permits in principle to start the formulation of a scattering theory (with a tremendous lot of mathematical rigor problems).

Note that, owing to the difficulty of proving convergence in (2.10), (2.13), an alternative system of axioms assuming first the existence of an evolution operator U satisfying eq. (2.12) is more tractable.

But for the present time we consider the assumptions (a) (b) (c) (d) (e) (f) because they exhibit a close analogy with some already familiar features of N-body hamiltonian dynamics.

We can obviously construct an example of (a) (b) (d) which is not trivial in the sense that $\mathcal{N}$ is not conserved:

Let B be any operator unitary in the rigged Hilbert space and such that $[B, \mathcal{N}] \neq 0$.

Then (a) (b) (d) hold true for

$$h_a^N = B \cdot \bar{h}_a^N \cdot B^{-1} \tag{3.1}$$

where $\bar{h}^N_a$ are the free particle generators.

In that case it is easy to legitimate the formulas $M = B\,\bar{M}\,B^{-1}$, $U = B\,\bar{U}\,B^{-1}$ where $\bar{M}$ and $\bar{U}$ are respectively the mass and evolution operators for free particles. Then B maps every element of K_m

onto an element of J'_m.
It is not easy to choose B in order to satisfy (c) non trivally.
From a constructive point of view, the first serious task will be finding such an operator.
This question is left for further investigation, but we already understand that the breaking of particle number conservation will involve creation and annihilation operators in $\mathcal{H}$.
Fortunately the transformation properties of these operators under Poincaré group are well-known from the general theory of Hilbert spaces.

This could lead us to employ an object similar to the free field operator of conventional Q.F.T. with respect to its covariance properties and that we call the "off-shell field operator" [18].

It is different from its Q.F.T. counterpart because, roughly speaking we work in $\mathcal{H}$ endowed with $\langle\ ,\ \rangle$ whereas Q.F.T. stands within the mass-shell endowed with $(\ ,\ )$.
We expect that it will be the right ingredient for constructing B as we wish, since we control its properties under Poincaré transformations.

For the moment we remain with this program to undertake and various other problems to investigate.

For example: construct h^N_a by a trick more general than eq. (3.1).

Define the physical scalar product $(\ ,\ )$ on the mass-shell space etc...

Before we go further it might be neccesary to look back and achieve to clarify many points in N-body dynamics.

At least we have attempted to open a window on a domain which could be of great interest in the future.

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In the constraint formalism it may occur that the constraint operators do not commute strongly. In that case L^2 is replaced by a subspace in which their commutators vanish. Within this reduced space the situation is algebraically similar to the one we consider.

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For the sake of simplicity and convenience we introduce hermiticity in the rigged Hilbert space with the implicit assumption that the domains of the operators include $\mathcal{S}$.
A more accurate and general treatment would involve the concept of self-adjointness in the rigged Hilbert space in the sense of the above reference.
The existence of realistic interacting hamiltonians which map $\mathcal{S}$ into itself is by no means obvious. In fact, the way we rig $L^2(\mathbb{R}^{4N})$ might be too restrictive for practical purposes. For instance an alternative choice $\mathcal{D} \subset L^2 \subset \mathcal{D}'$ (where $\mathcal{D}$ is the space of C^∞ functions with compact support) would perhaps give us more freedom.
But $\mathcal{S}$ behaves more simply than $\mathcal{D}$ under Fourier transformation. Anyway, for the simplicity of the discussion we choose $\mathcal{S}$ as the nuclear subspace in the present picture. This choice could be modified if necessary.

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As usually we have:
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THE ORIGINS OF PREDICTIVE RELATIVISTIC MECHANICS

Robert Nyden Hill

Department of Physics
University of Delaware
Newark, Delaware 19711

It is an honor to be invited here to speak to you about the origins of a field to which I contributed a number of years ago. It is gratifying to see how it has grown in the intervening years -to the point where a conference such as this can now be held- a conference which will undoubtedly stimulate still further interest and growth.

Because I have not been active in relativistic dynamics for almost ten years, I will speak mostly of things which happened ten or more years ago, and leave the task of describing more recent developments to others. Much of what I will cover is described in more detail in the 1972 reprint collection edited by Edward H, Kerner, entitled "The Theory of-Action-at-a-Distance in Relativistic Particle Dynamics"[1]. I will begin by outlining the various forms of action-at-a-distance relativistic dynamics. A more detailed discussion of predictive relativistic mechanics will follow. Finally, because I always like to present at least one thing which is new, I will discuss a simple example, drawn from electromagnetic theory, for which the predictive point of view appears to fail.

The action-at-a-distance point of view was predominant from the time of Newton to the time of Maxwell and Einstein. The time since Maxwell has, however, been an age of field theory, in which the action-at-a-distance point of view has been largely ignored. Recent years have seen a reawakening of action-at-a-distance.

Two distinct threads can be seen in this reawakening. The first starts with Schwarzschild[2], Tetrode[3], and Fokker[4], and runs through Dirac[5] to the electrodynamics of Wheeler and Feynman[6], which is itself a special case of the later relativistic mechanics of van Dam and Wigner[7,8]. This thread is marked by manifest covariant, many-time

theories. The forces between particles act along light cones, or, in the case of van Dam and Wigner, through the space-like region between the past and future light cones. Equations of motion are coupled differential-difference equations. For Wheeler-Feynman electrodynamics, at least, this differential-difference structure can be viewed as a vestige of field theory which has not been removed: it arises because of the finite time required for the electromagnetic field to propagate from one particle to another.

The other thread, to which predictive relativistic mechanics belongs, starts with a 1949 paper of Dirac[11]. It is marked by equations of motion which are coupled ordinary differential equations. In Dirac's instant form -which is the form which has been picked up and developed- the action-at-a-distance is then instantaneous- as in ordinary non-relativistic Newtonian mechanics. The theory is a single-time theory, rather than a many-time theory.

I will first review briefly the manifest covariant many-time theories of Wheeler-Feynman and Van Dam Wigner, and will then discuss predictive relativistic mechanics in considerably greater detail. Wheeler-Feynman electrodynamics is characterized by the action principle

$$J = -\sum_a m_a c \int (-da_\mu \, da^\mu)^{1/2} + \\ + \sum_{a<b} \frac{e_a e_b}{c} \iint \delta[(a_\mu - b_\mu)(a^\mu - b^\mu)] \, da_\nu \, db^\nu \\ = \text{extremum} \qquad (1)$$

Here a_μ and b_μ are the four-vector space-time coordinates of particles a and b. The equations of motion which follow from this action principle are

$$m_a c^2 \ddot{a}_\sigma = e_a \sum_{b \neq a} F^{(b)}_{\sigma\rho}(a) \cdot \dot{a}^\rho \qquad (2)$$

where a dot denotes differentiation with respect to proper time. The antisymmetric electromagnetic field four-tensor $F_{\sigma\rho}{}^{(b)}(a)$ describing the field at particle a due to particle b is the sum

$$F^{(b)}_{\sigma\rho}(a) = \frac{1}{2}\left[F^{(b)}_{\sigma\rho\,\mathrm{adv}}(a) + F^{(b)}_{\sigma\rho\,\mathrm{ret}}(a) \right] \qquad (3)$$

of advanced and retarded contributions; interactions take place along both past and future light cones, as shown in figure 1. Thus the

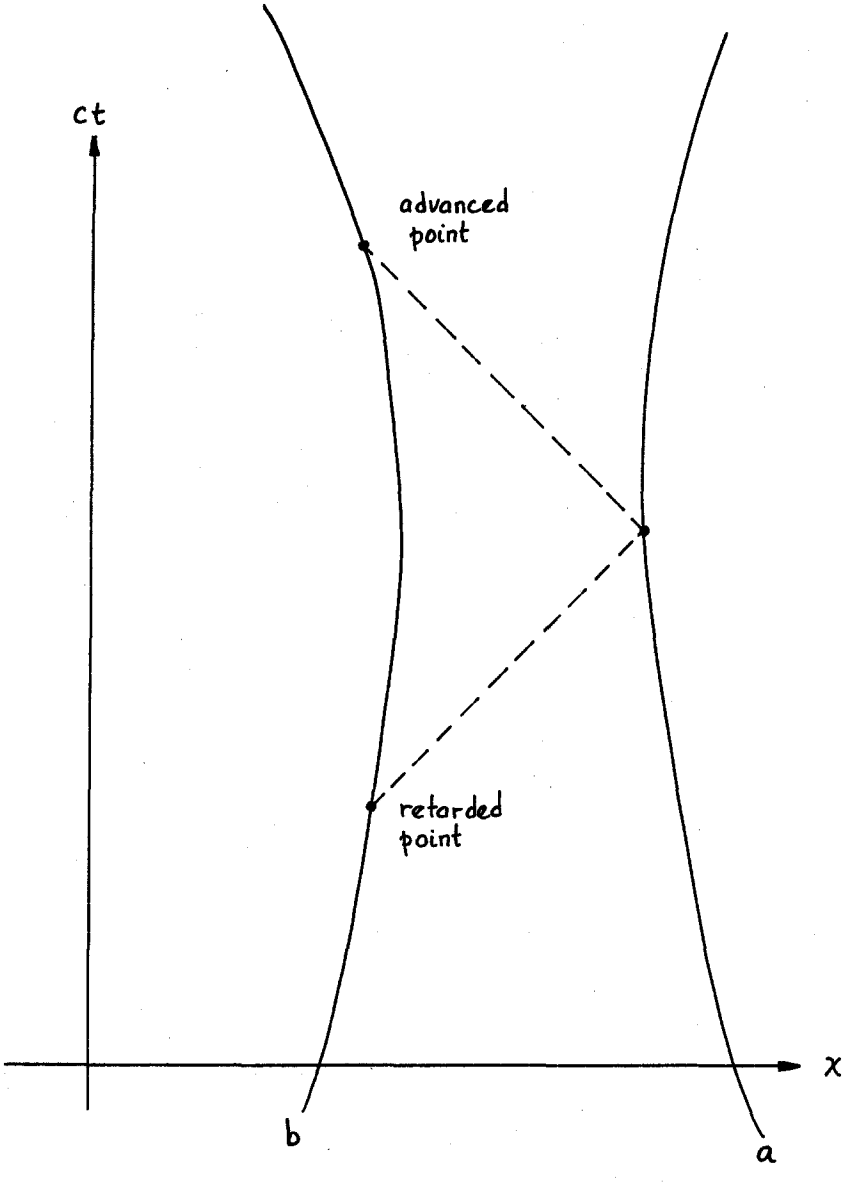

Figure 1. Wheeler-Feynman electrodynamics .

equations of motion have a differential-difference structure: computation of the force on particle a requires a knowledge of the positions and velocities of all other particles at both the advanced and retarded times. The mathematical theory of equations such as this is not in good shape. For linear differential-difference equations, specification of initial data along entire segments of the solution curves is needed to guarantee existence and uniqueness of solutions. The equations of motion of Wheeler-Feynman electrodynamics, however, are nonlinear differential-difference equations. The only rigorous results for these of which I am aware are a theorem of Driver for two particles in one dimension[9], which shows that Newtonian initial data (specification of positions and velocities at one instant of time) can be enough to guarantee existence and uniqueness. The Van Dam-Wigner mechanics can be viewed as a generalization of Wheeler-Feynman electrodynamics in which the force on particle a due to particle b is determined by the position and velocity of particle b along the entire segment of particle b's world line between the past and future light cones.

Conserved quantities in these theories are sums of the usual free-particle expressions plus an interaction contribution. The conserved energy momentum four vector P_μ and the angular momentum-center-of-mass momentum four tensor $M_{\mu\nu}$ have the forms

$$P_\mu = \sum_a m_a \dot{a}_\mu + P'_\mu \tag{4}$$

$$M_{\mu\nu} = \sum_a m_a (a_\mu \dot{a}_\nu - a_\nu \dot{a}_\mu) + M'_{\mu\nu} \tag{5}$$

where P'_μ and $M'_{\mu\nu}$ are the interaction contributions. Those interactions contributions are expressible as integrals over the world lines; physically they represent energy-momentum (or angular momentum-center-of-mass momentum) in transit, which has left one particle but not yet arrived at another particle. The P'_μ are expected to vanish asymptotically when all of the particles are well separated. However, as was first pointed out by Van Dam and Wigner[8], $M'_{\mu\nu}$ need not vanish asymptotically for long-range forces such as those of electrodynamics. The reason is fairly simple once it has been pointed out. The interaction energy-momentum P'_μ falls off like (interparticle distance)$^{-1}$ for forces which fall off like (interparticle distance)$^{-2}$. When the momentum is multiplied by a distance to form an angular momentum, one has interaction contributions of the form (interparticle distance)$^{-1}$ x (distance) which need not -and in general do not- vanish asymptotically. Interaction contributions $M'_{\mu\nu}$ which do not vanish asymptotically for long range forces also appear in single-time instantaneous action-

at-a-distance versions of relativistic dynamics.

The reconciliation of the half-advanced half-retarded interactions of Wheeler-Feynman electrodynamics with the usual retarded interactions plus radiation damping follows from the absorber condition. Dirac[5] showed that the radiation damping force on particle a was derivable from a field $F_{\mu\nu\,\mathrm{rad}}(a)$, which could be obtained as the limit as the source point approaches the field point of $\frac{1}{2}\left[F^{(a)}_{\mu\nu\,\mathrm{ret}} - F^{(a)}_{\mu\nu\,\mathrm{adv}}\right]$. As a consequence,

$$\sum_{\substack{b\\ b\neq a}} F^{(b)}_{\mu\nu\,\mathrm{ret}}(a) + F_{\mu\nu\,\mathrm{rad}}(a) =$$

$$= \sum_{\substack{b\\ b\neq a}} \tfrac{1}{2}\left[F^{(b)}_{\mu\nu\,\mathrm{adv}}(a) + F^{(b)}_{\mu\nu\,\mathrm{ret}}(a)\right] +$$

$$+ \sum_{b} \tfrac{1}{2}\left[F^{(b)}_{\mu\nu\,\mathrm{ret}}(a) - F^{(b)}_{\mu\nu\,\mathrm{adv}}(a)\right] \qquad (6)$$

The absorber condition is that the field $\sum_b (F^{(b)}_{\mu\nu\,\mathrm{ret}} - F^{(b)}_{\mu\nu\,\mathrm{adv}})$ vanish; since the retarded and advanced fields have the same sources, this field obeys free field equations and hence vanishes for all time if it and its time derivative vanish at one instant of time. A generalization of Wheeler-Feynman electrodynamics in which the retarded field minus the advanced field need not vanish has been given by Rohrlich[10].

I will now take up the instantaneous action-at-a-distance theories. Key ideas in this approach, which I will discuss in turn, are (1) a Hamiltonian formulation in which the Poincaré group is canonically represented, (2) world line invariance, (3) the often unremarked tacit assumption that physical coordinates are canonical coordinates, and (4) the assumption that the given equations of motion hold for all time, from past infinity to future infinity.

Key idea (1), the requirement of a Hamiltonian formulation, was motivated by the desire to obtain a quantum theory via the conventional route of quantizing a classical Hamiltonian theory. Relativistic corrections to non-relativistic Hamiltonian theories can be conveniently treated this way. In atomic physics, for example, the non-relativistic problem is already difficult to solve to the desired accuracy; thus one would like to treat relativistic corrections within the same Hamiltonian framework so as to take advantage of the hard-won understanding of the non-relativistic problem. A relativistic Hamiltonian mechanics, as outlined by Dirac, consists of a set of ten generators satisfying the Poisson bracket algebra of the Poincaré group. If we

assign the letters H, $\vec{P}$, $\vec{J}$, $\vec{K}$ to the generators of time translations, space translations, spatial rotations, and pure Lorentz transformations, then generators must satisfy, in dyadic notation with $\vec{\vec{I}}$ the identity dyadic,

$$[\vec{P}, \vec{P}] = 0 \tag{7a}$$

$$[\vec{P}, H] = 0 \tag{7b}$$

$$[\vec{J}, \vec{J}] = -\vec{\vec{I}} \times \vec{J} \tag{7c}$$

$$[H, \vec{J}] = 0 \tag{7d}$$

$$[\vec{P}, \vec{J}] = -\vec{\vec{I}} \times \vec{P} \tag{7e}$$

$$[\vec{K}, \vec{K}] = \vec{\vec{I}} \times \vec{J} \tag{7f}$$

$$[H, \vec{K}] = -\vec{P} \tag{7h}$$

$$[\vec{P}, \vec{K}] = -\vec{\vec{I}} H \tag{7i}$$

and

$$[\vec{J}, \vec{K}] = -\vec{\vec{I}} \times \vec{K} \tag{7j}$$

From this point of view, the problem of constructing a relativistic dynamics is seen as a problem of finding a set of generators H, $\vec{P}$, $\vec{J}$, $\vec{K}$, depending on canonical coordinates and momenta, which satisfy the Poisson bracket relations (1a) - (1j) .

Key idea (2), the notion of world line invariance, applies only to point particles. Stated most simply, it is the requirement that observers in different inertial frames agree as to what are -and what are not- allowed sets of particle world lines. To make this more explicit with a concrete example, refer to figure 2 which shows a pair of world lines for two particles moving in one dimension. Suppose that the equations of motion are second order ordinary differential equations giving particle accelerations as functions of positions and velocities, with enough regularity so that specification of Newtonian initial data -i.e., specification of initial positions and velocities- is sufficient to guarantee existence and uniqueness of the solution curves, which are the sets of world lines. Then an observer in frame S , using space-time coordinates (x,ct) , would specify initial positions and velocities at points P_o and Q_o and obtain the world

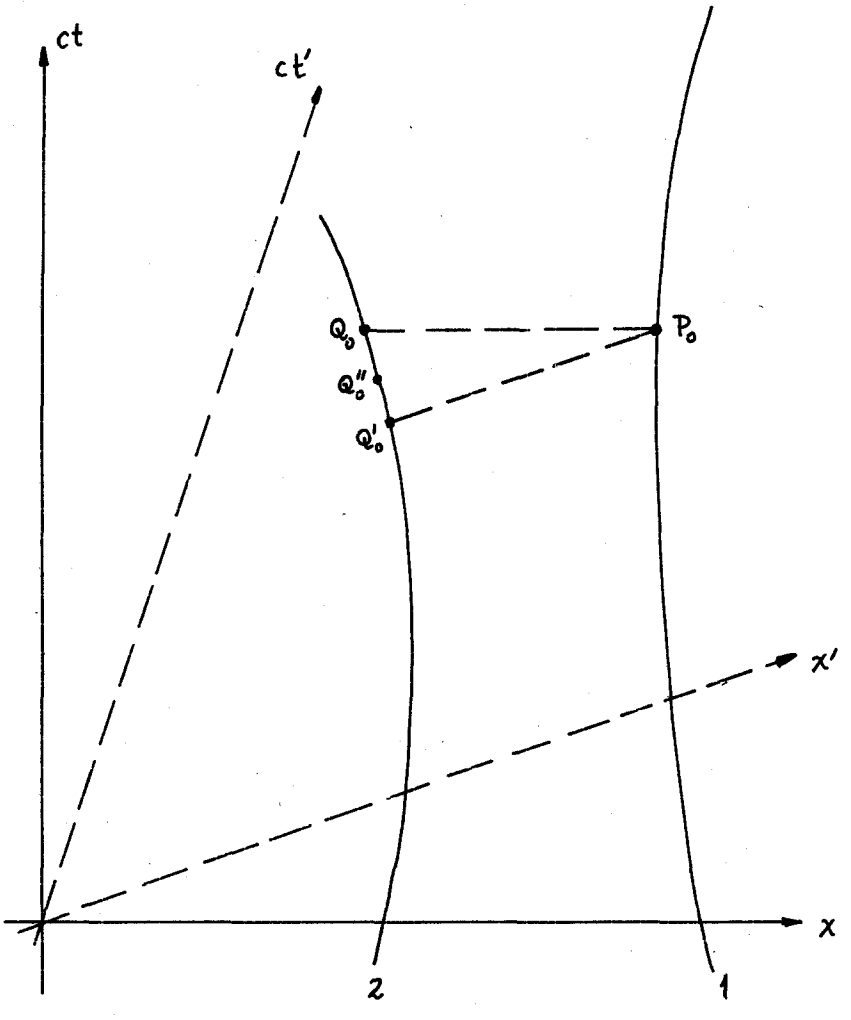

Figure 2 . World line Invariance .

lines shown in figure 2. An observer in frame S', using space time coordinates (x', ct') , would specify initial position and velocities at points P_o and Q_o' . Let the initial positions and velocities at points P_o and Q_o' be the Lorentz transforms from S to S' of the

the other three ideas. Interaction is possible, but the canonical particle coordinates are then not the coordinates of point particles. The work of Thomas[17], of Bakamjian and Thomas[18], and of Foldy[19] belongs to this branch. The other branch gives up the Hamiltonian scheme initially, opting instead for a Newtonian format in which particle accelerations are given as functions of positions and velocities. Douglas Currie, Edward Kerner, and I started this approach, which was not manifestly covariant in its original formulation. A Hamiltonian scheme for these Newtonian equations of motion can be obtained via the Lie-Konigs theorem, as suggested originally by Kerner[20], by giving up the requirement that physical positions be canonical. This Hamiltonian scheme can be made unique up to canonical transformation by imposing the condition of asymptotic reduction to the usual free-particle form.

Because I was asked to speak about "the origins of predictive relativistic mechanics", it is perhaps appropriate that I describe how I became involved in this field. In 1962-63 both Douglas Currie and I were postdoctoral fellows at Princeton University, where I heard Doug speak about the zero interaction theorem, which was at that time interpreted as proving that instantaneous action-at-a-distance was impossible in relativistic dynamics. I remember being skeptical of this interpretation even then, based on the following argument: Specification of initial positions and velocities is sufficient to determine a unique set of world lines in the non-relativistic limit. If relativistic corrections can be calculated by perturbing about the non-relativistic limit, then initial positions and velocities would be sufficient to determine a unique set of world lines in the relativistic case also, in the form

$$\vec{r}_i = \vec{\rho}_i \ (t;\ \text{initial conditions}). \qquad (8a)$$

By differentiation, one would have

$$\vec{v}_i = \partial\vec{\rho}_i/\partial t \qquad (8b)$$

and

$$\vec{a}_i = \partial^2\vec{\rho}_i/\partial t^2 \qquad (8c)$$

Equations (8a) and (8b) could in principle be solved for the initial conditions; if these solutions were inserted in (8c), one would have equations of motion of the form

$$\vec{a}_i = \vec{f}_i(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_n;\ \vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n) \qquad i = 1,2,3,\ldots,n$$

positions and velocities at P_o and Q_o' for the solutions found by the observer in S . Then world line invariance holds if the observer in S' , using this initial data, obtains world lines which are the Lorentz transforms of those obtained by the observer in S -i.e., which are the same as those found in S when drawn in a space-time diagram such as figure 2. The notion of world line invariance is, of course, not confined to equations of motion which are ordinary differential equations- it holds, for example, in the ordinary electrodynamics of point particles, in Wheeler-Feynman electrodynamics, and in the mechanics of van Dam and Wigner. All that matters is that different observers obtain sets of world lines which are the same from a physical point of view.

On the other hand, world line invariance need not hold for particles which are not point particles. The coordinate of a particle with structure -which can be thought of as a generalization of the non-relativistic notion of a center of mass- is some kind of an average over that structure. Computation of such averages in different Lorentz frames need not yield physically identical world lines, as pointed out in an illuminating discussion by Gordon Fleming[12] . Thus lack of world line invariance is not a fatal flaw -it merely means that the particles are not point particles.

The third key idea is very simple -it is the often unremarked tacit assumption that the physical coordinates of point particles, satisfying the world line invariance condition, are the canonical coordinates of the Hamiltonian formulation. The fourth key idea is equally simple -the equations of motion include <u>all</u> forces which act on the particles- the given equations of motion are the equations of motion for all time. In the case of instantaneous action-at-a-distance with point particles, for which world line invariance must hold, this implies that local external perturbations are not allowed. In figure 2, if I interfere with the motion of particle 2 at world point Q_o'' , predictions made by the observers in S and S' will not in general agree.

I have spelled out all of these key ideas explicitly because there is a certain incompatibility among them. The famous zero interaction theorem of Currie, Jordan, and Sudarshan[13] , proved originally for 2 particles, extended to 3 particles by Cannon and Jordan[14], to N particles by Leutwyler[15], and clarified in one and two spatial dimensions by myself[16], shows that all four of these key ideas can hold only for a free particle dynamics; at least one of them must be given up if the particles are to interact. The development splits into two branches at this point. One branch gives up world line invariance, but maintains

for an n-particle system -i.e., instantaneous action-at-a-distance! I recall arguing in this vein with Currie, but neither of us could convince the other and it was forgotten until I took my first job as a young assistant professor at the University of Delaware in the fall of 1964. There I met Ed Kerner, who was working with action-at-a-distance ideas. He explained his ideas to me -he was working out a systematic reduction of Wheeler-Feynman electrodynamics to instantaneous action-at-a-distance form, order by order in powers of $1/c^2$. Kerner's work was motivated by the fact that Wheeler-Feynman electrodynamics did not have the self-energy divergences which have plagued both classical and quantum field theories; he hoped to obtain a Hamiltonian formulation of classical mechanics via the Lie-Konigs theorem, which could be quantized via the usual Poisson bracket to commutator prescription to obtain a quantum electrodynamics which would at least be free of divergent self-energies.

I remember being struck by the fact that relativity played no role in his computations. The electrodynamics from which he started was certainly relativistically invariant; there must be some vestige of this invariance in the approximate equations of motion he was deriving. I puzzled over this for some time; the ultimate result was my first paper on this subject[21], which appeared in 1967 after a prolonged exchange with a skeptical referee (the paper was originally submitted in November of 1965). This paper contained, among other things, my derivation of what have come to be known as the Currie-Hill conditions. These same conditions were derived independently by Currie, whose paper[22] was submitted about a month before mine. Because these conditions have played an important role, I will sketch my original derivation of them for the special case of two particles in one dimension. In the notation of Figure 3, Q_0 and P_0 are simultaneous in the inertial frame S ; Q_0' and P_0 are simultaneous in S' . Thus, an observer in S computes the acceleration at P_0 in terms of coordinates and velocities at Q_0 and P_0 , while an observer in S' computes the acceleration at P_0 in terms of coordinates and velocities at Q_0' and P_0 . We require that the two observer's expressions for the acceleration at P_0 , written as

$$a_1(P_0) = f_1\left[x_1(P_0) - x_2(Q_0),\ v_1(P_0),\ v_2(Q_0)\right] \tag{9}$$

in S and as

$$a_1'(P_0) = f_1\left[x_1'(P_0) - x_2'(Q_0'),\ v_1'(P_0),\ v_2'(Q_0')\right] \tag{10}$$

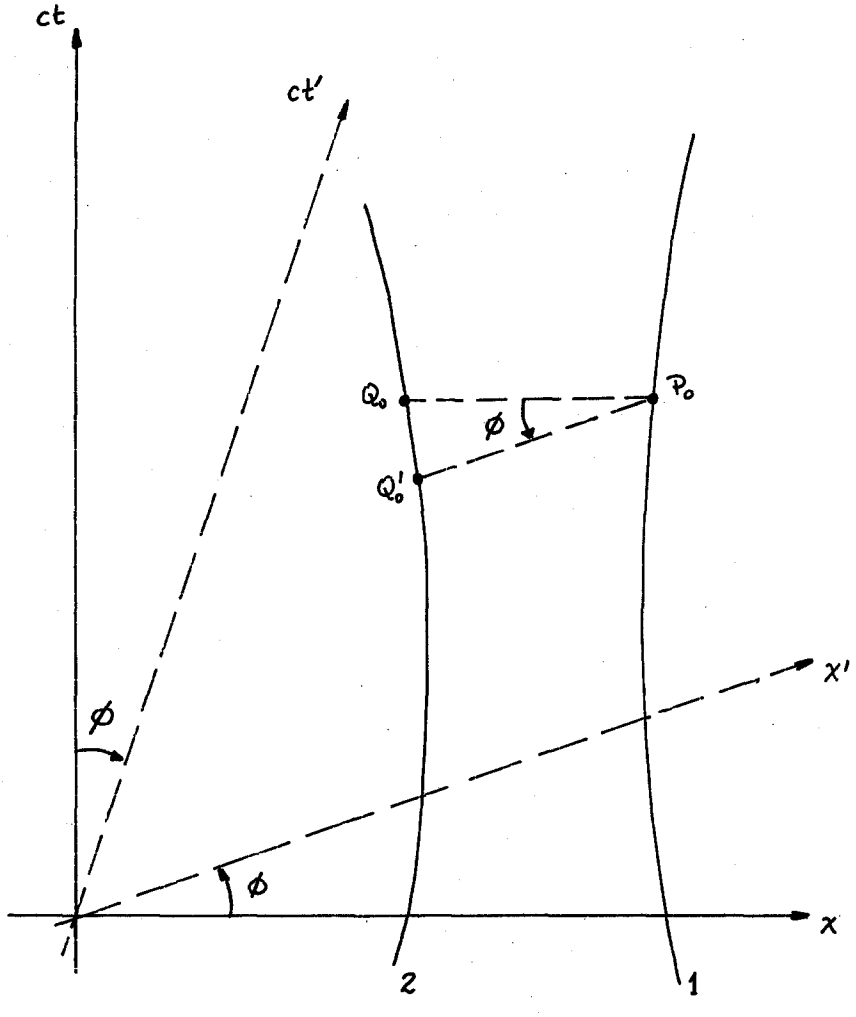

Figure 3. Instantaneous action-at-a-distance in different Lorentz frames. $z = \tan\phi$ is the relative velocity of S and S' .

in S' (same f_1 in both S and S), agree when one makes use of the Lorentz transformation formula for accelerations

$$a_1'(P_0) = (1-z^2)^{3/2}\cdot[1 - z v_1(P_0)]\cdot a_1(P_0) \tag{11}$$

to relate the numbers assigned to the acceleration at P_0 by the two observers.

This requirement is most easily enforced by imposing it upon the infinitesimal Lorentz transformation. Inasmuch as it is (from the viewpoint of group theory) the requirement that the infinitesimal transformation on the form of the equations of motion vanish, the group property of the Lorentz transformations then guarantees that it holds for all proper Lorentz transformations. We now think of S' as moving with respect to S with an infinitesimal velocity $z = \delta\beta$. Then $x' = x - t\cdot\delta\beta$ and $t' = t - x\cdot\delta\beta$ are the kinematical transformations; $t(Q_0') - t(Q_0) = -\left[x_1(P_0) - x_2(Q_0)\right]\cdot\delta\beta$ takes account of the change in simultaneity. Using these

$$x_1'(P_0) = x_1(P_0) - t_0\cdot\delta\beta$$

$$x_2'(Q_0') = \underbrace{x_2(Q_0) - t_0\cdot\delta\beta}_{\substack{\text{kinematical}\\ \text{Lorentz}\\ \text{transformation}}} - \underbrace{v_2(Q_0)\cdot\left[x_1(P_0) - x_2(Q_0)\right]\cdot\delta\beta}_{\substack{\text{world-point shift}\\ \text{from } Q_0 \text{ to } Q_0'}}$$

$$v_1'(P_0) = v_1(P_0) - \left(1 - v_1^2(P_0)\right)\cdot\delta\beta \tag{12}$$

$$v_2'(Q_0') = \underbrace{v_2(Q_0) - \left[1 - v_2^2(Q_0)\right]\cdot\delta\beta}_{\substack{\text{kinematical Lorentz}\\ \text{transformation}}} - \underbrace{a_2(Q_0)\cdot\left[x_1(P_0) - x_2(Q_0)\right]\cdot\delta\beta}_{\substack{\text{world-point shift}\\ \text{from } Q_0 \text{ to } Q_0'}}$$

$$a_1'(P_0) = a_1(P_0) + 3\,a_1(P_0)\cdot v_1(P_0)\cdot\delta\beta$$

If we insert the transformations (12) into (10), expand to first order in $\delta\beta$, and demand agreement with (9) to first order, we obtain the condition

$$3 a_1 v_1 = x_{12}\cdot v_2\cdot(\partial f_{11}/\partial x_{12}) - (1-v_1^2)\cdot(\partial f_{11}/\partial v_1) -$$

$$- (1 - v_2^2 + x_{12}\, a_2)\cdot(\partial f_{11}/\partial v_2)$$

We now use $a_1 = f_1$ and assume that the acceleration of particle 2

has been written in the instantaneous action-at-a-distance form, $a_2 = f_2(x_{12}, v_1, v_2)$. We then arrive at the differential statements of the Lorentz invariance of a one-dimensional two-body instantaneous action-at-a-distance theory

$$3 v_1 f_1 + L f_1 = - x_{12} \cdot f_2 \, (\partial f_1 / \partial v_2) \tag{13a}$$

$$3 v_2 f_2 + \tilde{L} f_2 = x_{12} \cdot f_1 \cdot (\partial f_2 / \partial v_1) \tag{13b}$$

where the linear first-order differential operators L and $\tilde{L}$ are defined by

$$L \equiv - x_{12} v_2 \, (\partial / \partial x_{12}) + (1 - v_1^2) \cdot (\partial / \partial v_1) + \tag{14a}$$

$$+ (1 - v_2^2) \, (\partial / \partial v_2)$$

$$\tilde{L} \equiv - x_{12} \cdot v_1 \cdot (\partial / \partial x_{12}) + (1 - v_1^2) \cdot (\partial / \partial v_1) + \tag{14b}$$

$$+ (1 - v_2^2) \cdot (\partial / \partial v_2)$$

The generalization of these conditions to N particles and 3 dimensions is straightforward. The conditions are quite general, and apply to <u>any</u> relativistic instantaneous action-at-a-distance theory in which the particles are point particles. Unfortunately they are non-linear, which makes the task of writing down particular solutions rather difficult. The general solution of (13)-(14) has, however, been found in implicit form[23]. This general solution is characterized by two arbitrary functions $f(\xi, \zeta)$ and $g(\eta, \zeta)$. With $\alpha(\xi, \eta, \zeta)$ and $\beta(\xi, \eta, \zeta)$ defined by

$$\alpha(\xi, \eta, \zeta) = f(\xi, \zeta) - \xi \cdot \frac{\partial f(\xi, \zeta)}{\partial \xi} + g(\eta, \zeta) - \eta \, \frac{\partial g(\eta, \zeta)}{\partial \eta} \tag{15a}$$

and

$$\beta(\xi, \eta, \zeta) = \frac{\partial f(\xi, \zeta)}{\partial \xi} + \frac{\partial g(\eta, \zeta)}{\partial \eta} \tag{15b}$$

we have

$$x_{12} \equiv x_1 - x_2 = (\alpha \beta)^{1/2} \tag{16a}$$

$$v_1 = (\alpha - \beta \xi) / (\alpha + \beta \xi) \tag{16b}$$

$$v_2 = (\alpha - \beta\eta)/(\alpha + \beta\eta) \tag{16c}$$

Equations (15)-(16) define a transformation from the physical variables x_{12}, v_1, v_2 to new variables ξ, η, ζ. The solutions of (13)-(14) are then

$$f_1(x_{12}, v_1, v_2) = -4(\alpha\beta)^{3/2}(\alpha+\beta\xi)^{-3}\cdot(\partial^2 f/\partial\xi^2)^{-1} \tag{17a}$$

and

$$f_2(x_{12}, v_1, v_2) = 4(\alpha\beta)^{3/2}(\alpha+\beta\eta)^{-3}\cdot(\partial^2 g/\partial\eta^2)^{-1} \tag{17b}$$

The equations of motion (17) can actually be integrated to obtain the world lines in parametric form; the result is

$$x_1 = \frac{1}{2}\Big\{ c_1 + c_2 + \Phi^{1/2}\cdot\frac{\partial f(\xi,\zeta)}{\partial\xi} + \Phi^{-1/2}\cdot\Big[f(\xi,\zeta) - \xi\cdot\frac{\partial f(\xi,\zeta)}{\partial\xi}\Big]\Big\} \tag{18a}$$

$$t = \frac{1}{2}\Big\{ c_1 - c_2 + \Phi^{1/2}\cdot\frac{\partial f(\xi,\zeta)}{\partial\xi} - \Phi^{-1/2}\cdot\Big[f(\xi,\zeta) - \xi\cdot\frac{\partial f(\xi,\zeta)}{\partial\xi}\Big]\Big\} \tag{18b}$$

$$x_2 = \frac{1}{2}\Big\{ c_1 + c_2 - \Phi^{1/2}\cdot\frac{\partial g(\eta,\zeta)}{\partial\eta} - \Phi^{-1/2}\cdot\Big[g(\eta,\zeta) - \eta\cdot\frac{\partial g(\eta,\zeta)}{\partial\eta}\Big]\Big\} \tag{18c}$$

$$t = \frac{1}{2}\Big\{ c_1 - c_2 - \Phi^{1/2}\cdot\frac{\partial g(\eta,\zeta)}{\partial\eta} + \Phi^{-1/2}\cdot\Big[g(\eta,\zeta) - \eta\cdot\frac{\partial g(\eta,\zeta)}{\partial\eta}\Big]\Big\} \tag{18d}$$

The constants of the motion are c_1, c_2, Φ and ζ. Equations (18a) and (18b) give the world line of particle 1 with ξ as parameter; similarly (18c) and (18d) give the world line of particle 2 with η as parameter. The choice of Φ fixes the Lorentz frame: equating the expressions (18b) and (18d) for t yields the relation between the parameters ξ and η in the form

$$\alpha(\xi,\eta,\zeta) = \Phi\cdot\beta(\xi,\eta,\zeta) \tag{19}$$

Implicit general solutions such as the one found above are, unfortunately, not as useful as explicit solutions. Some explicit closed form solutions have been found, including the very amusing example

$$-a_1 = a_2 = \frac{(v_1 - v_2)^2}{2(x_1 - x_2)} \tag{20}$$

which was discovered independently by Kerner[24] and by Currie. The example (20) is actually both Galilean and Lorentz invariant! There are no limits on particle velocities; the velocity of light c does not enter. Galilean invariance is obvious; Lorentz invariance can be verified by showing that (13) and (14) are satisfied, or by writing down the solutions, which are the parabolas,

$$x_1 = A + Bt + (|Ct + D|)^{1/2} \tag{21a}$$

$$x_2 = A + Bt - (|Ct + D|)^{1/2} \tag{21b}$$

and showing explicitly that the Lorentz transformation carries solutions into (different) solutions: For the world line of particle 1, the Lorentz transformation

$$x_1 = \frac{x_1' + vt'}{\sqrt{1 - (v/c)^2}} \tag{22a}$$

$$t = \frac{t' + vx_1'/c^2}{\sqrt{1 - (v/c)^2}} \tag{22b}$$

carries (21a) into

$$\frac{x_1' + vt'}{\sqrt{1 - (v/c)^2}} = A + B\,\frac{t' + vx_1'/c^2}{\sqrt{1 - (v/c)^2}} + \sqrt{\left| C\,\frac{(t' + vx_1'/c^2)}{\sqrt{1 - (v/c)^2}} + D \right|}$$

which can be solved for x_1' to obtain

$$x_1' = A' + B't' + \sqrt{|C't' + D'|} \tag{23}$$

with

$$A' = \frac{\sqrt{1-(v/c)^2}}{1+B(v/c)} \left\{ A - \frac{C(v/c)}{2[1+B(v/c)]} \right\} \tag{24a}$$

$$B' = \frac{B + (v/c)}{1 + B(v/c)} \tag{24b}$$

$$C' = [1-(v/c)^2]^{3/2} \cdot [1+B(v/c)]^3 C \tag{24c}$$

$$D' = [1-(v/c)^2] \cdot [1+B(v/c)]^{-2} \left\{ D - \right.$$

$$- AC(v/c)[1+B(v/c)]^{-1} +$$

$$\left. + \frac{1}{4}(v/c)^2 C^2 [1+B(v/c)]^{-2} \right\} \tag{24d}$$

A similar computation for the world line of particle 2 shows that the Lorentz transformation

$$x_2 = \frac{x_2' + v t'}{\sqrt{1-(v/c)^2}} \tag{25a}$$

$$t = \frac{t' + v x_2'/c^2}{\sqrt{1-(v/c)^2}} \tag{25b}$$

carries (21b) into

$$x_2' = A' + B' t' - \sqrt{|C' t' + D'|} \tag{26}$$

with the same A', B', C', D'. Actually, the dynamics (20) has the general linear group on x and t (which has the Poincaré group and the Galilei group as subgroups) as invariance group; it is known that transformations of the general linear group carry parabolas into parabolas.

The above example, although it shows that relativistically invariant instantaneous action-at-a-distance dynamics is possible, possesses two unphysical features: it is one dimensional, and it does not exhibit free particle motion at infinity. Unfortunately it seems very hard to find physically more realistic examples in explicit closed form. For electrodynamics, which was always the touchstone for Ed Kerner and myself, the best that thas been done as far as I know is an expansion to low order in powers of e^2. The convergence of this series has not to my knowledge been proven; furthermore truncation of this

series at some finite order leads to an only approximately relativistically invariant dynamics. The difficulty arises from the fact that classical electrodynamics contains only a mass m, a charge e, and the velocity of light c. No dimensionless combination of these is possible; expansions in powers of e^2 turn out to be expansions in powers of the classical electron radius $e^2/(mc^2)$ divided by an interparticle separation. The closest thing to a closed-form expression for the equations of motion of instantaneous action-at-a-distance electrodynamics that I am aware of is the set of coupled integrodifferential equations, given in my initial paper[21], which characterize the equations of motion for the two particle case.

The fact that we have given up the Hamiltonian formulation becomes a problem when we want to construct the corresponding quantum theory. A way around this difficulty was suggested by Kerner[20], who proposed that the action-at-a-distance dynamics in Newtonian format could be cast into Hamiltonian form by invoking the Lie-Konigs theorem. The Currie-Jordan-Sudarshan zero interaction theorem is circumvented by giving up the idea that physical positions are canonical variables in the Hamiltonian scheme.

The approach to Hamiltonian dynamics via the Lie-Konigs theorem[16,20] applies to any set of second-order differential equations $\vec{a}_i = \vec{F}_i(\vec{x}_1, \ldots, \vec{x}_N; \vec{v}_1, \ldots, \vec{v}_N; t)$ specifying particle accelerations as functions of position, velocity, and time. These are first rewritten as the first-order system

$$\frac{dy_i}{dy_0} = h_i(y_0, y_1, \ldots, y_{2N}) \quad ; \quad i = 1, 2, \ldots, 2N \tag{27}$$

Here we have in mind that $y_0 = t$, $y_i = x_i$, $h_i = v_i = y_{i+N}$, and $h_{i+N} = F_i(y_1, \ldots, y_N; y_{N+1}, \ldots, y_{2N}; y_0)$ for $i = 1, \ldots, N$. We seek to derive Eqs. (27) from a variational principle of the form

$$\delta \int \left[\sum_{i=1}^{2N} U_i(dy_i/dy_0) + U_0 \right] dy_0 = 0 \tag{28}$$

wherein the $y_i(y_0)$, $i = 1, \ldots, 2N$, are to be independently varied. The Euler equations of (12) take the form

$$\sum_{j=0}^{2N} \Gamma_{ij} (dy_j/dy_0) = 0$$

where the matrix Γ_{ij} is defined by

$$\Gamma_{ij} = (\partial U_i/\partial y_j) - (\partial U_j/\partial y_i) \tag{29}$$

In order that it be possible to solve the Euler equations for the derivatives (dy_j/dy_o), the $2N \times 2N$ matrix Γ_{ij} (with $i \neq 0$, $j \neq 0$) must be nonsingular. Solving these Euler equations for dy_j/dy_o yields the specified equations (27) if the U_i satisfy the differential conditions

$$\sum_{j=0}^{2N} \Gamma_{ij} h_j = 0 \quad , \quad i = 0, 1, \ldots, 2N \tag{30}$$

Here we have allowed $i = 0$ because the equation for $i = 0$ is a consequence of the other $2N$ equations.

Once Eqs. (30) have been solved to yield a set of U_i with nonsingular $\Gamma_{ij} (i \neq 0, j \neq 0)$, and hence a variational principle of the form (28), a Hamiltonian formulation can be obtained by solving Pfaff's problem to reduce the differential form $\sum_{i=1}^{2N} U_i dy_i$ to $\sum_{i=1}^{N} P_k dQ_k$ (the fact that this can always be done is the principal result of Pfaff's classic memoir). The Q_k and P_k are the canonical coordinates and momenta. The Hamiltonian is $H = -U_o$, and must be reexpressed in terms of the canonical variables P_k, Q_k obtained by solving Pfaff's problem.

Casting the dynamics into Hamiltonian form is, however, not enough. The transformations of the Poincaré group must be canonical transformations in that Hamiltonian scheme. Canonically inequivalent Hamiltonian formulations lead to inequivalent quantum theories when standard quantization methods are applied[25]; thus equivalent observers must be related by canonical transformations if they are to have equivalent quantum theories. In general only a subgroup of the invariance group of the differential equations (27) with which we started will be canonically represented. Starting with a dynamics which is invariant under the Poincaré group does not guarantee that the Poincaré group will be canonically represented in the Hamiltonian scheme obtained via the procedure I have just outlined. Thus it is neccesary to look at the question of invariance in more detail.

Start with infinitesimal transformations

$$y_i \longrightarrow y_i' = y_i + \epsilon q_i^{\alpha}(y_o, y_1, \ldots, y_{2N}) \tag{31}$$

which leave the original differential equations (27) invariant, i.e. such that both (27) and

$$\frac{dy_i'}{dy_o'} = h_i(y_o', y_1', \ldots, y_{2N}') + O(\epsilon^2) \tag{32}$$

hold for small ε . Here the Greek α indexes the different transformations. Insert (31) in (32), expand to first order in ε , compare with (27), and demand that the coefficient of ε vanish. The result is the condition

$$L_\alpha h_i - D g_i^\alpha + h_i D g_o^\alpha = 0 \tag{33}$$

where

$$L_\alpha = \sum_{i=0}^{2N} g_i^\alpha \partial/\partial y_i \tag{34}$$

and

$$D = \sum_{i=0}^{2N} h_i \partial/\partial y_i \tag{35}$$

Equation (33) is the condition that the transformation (31) leave the differential system (27) invariant. On the other hand, the condition that the transformation (31) be a canonical transformation is that it changes the differential from $\sum_{i=0}^{2N} U_i dy_i$ which appears in the variational principle (28) by an exact differential -i.e., that

$$\sum_{i=0}^{2N} \left[U_i(y_0, \ldots, y_{2N})\, dy_i - U_i(y_0', \ldots, y_{2N}')\, dy_i' \right] =$$

$$= \varepsilon\, d\Omega_\alpha + O(\varepsilon^2) \tag{36}$$

for some Ω_α . Inserting (31) in (36), expanding to first order in ε , equating coefficients of ε on both sides, and demanding that the result hold for arbitrary dy_i yields

$$L_\alpha U_i + \sum_{j=0}^{2N} U_j \left(\partial g_j^\alpha/\partial y_i\right) = \partial\Omega_\alpha/\partial y_i \tag{37}$$

as the condition that (31) be a canonical transformation in the Hamiltonian scheme. An equivalent condition[26], which can be used to test whether or not a given transformation is canonical, is

$$L_\alpha \Gamma_{jk} + \sum_{i=0}^{2N} \left[\Gamma_{ik} \left(\partial g_i^\alpha/\partial y_j\right) + \Gamma_{ji} \left(\partial g_i^\alpha/\partial y_k\right) \right] = 0 \tag{38}$$

It can be shown that (38) implies (33); however (33) does not imply (38). Make the definition

$$G_\alpha \equiv \sum_{i=0}^{2N} g_i^\alpha U_i - \Omega_\alpha \tag{39}$$

It can then be shown[26] that G_α is a conserved quantity; this is Noether's theorem in the present context. It can also be shown[26] that

$$y_\kappa + \varepsilon [y_\kappa, G_\alpha] = y_\kappa + \varepsilon (g^\alpha_\kappa - g^\alpha_0 h_\kappa) =$$

$$= y'_\kappa + (y_0 - y'_0) h_\kappa \qquad (40)$$

which holds for $i = 0,1,\ldots,2$ if h_0 is defined to be 1. Here the brackets $[.,.]$ are Poisson brackets. The result (40) shows that G_α generates the canonical transformation (31) with which it is associated <u>if</u> $g_0 = 0$; if $g_0 \neq 0$, the formalism compensates for the inability of the Poisson bracket to transform the time $t = y_0$ by shifting the other variables an amount $-\varepsilon g^\alpha_0 (dy_k/dy_0)$ along the solution curves of (27).

A deeper understanding of the result (40) can be had by taking a closer look at the group of invariance transformations of (27), which group we call G . The subgroup of G which is canonically represented will be denoted by G_c . There is a subgroup H of G , generated by transformations of the form

$$y_i \longrightarrow y'_i = y_i + \varepsilon \Phi(y_0, y_1, \ldots, y_{2N}) \cdot h_i \qquad (41)$$

where Φ is an arbitrary function, which leaves solution curves invariant -i.e., which carries a solution $y_j = f_j(y_0)$ of (27) into $y'_j = f_j(y'_0)$ with the same f_j . It can be shown[26] that H is an invariant (normal) subgroup of both G and G_c . Thus we can decompose G (or G_c) into cosets relative to H and consider the factor groups (quotient groups) G/H and G_c/H . Each element of a given coset has the same effect on a solution curve $y_j = f_j(y_0)$ of (27) . Thus if we identify physical states with the solution curves (i.e., think of the physical state as a state <u>sub-specie aeternatis</u> rather than an instantaneous state), it is the transformations of the factor groups G/H and G_c/H which change the physical state.

Each coset in the decomposition relative to H contains one and only one transformation which leaves the time $t = y_0$ fixed; thus the transformations which leave $t = y_0$ fixed provide faithful representations of the factor groups G/H and G_c/H which change physical states. In particular, the transformations generated by the usual Poisson brackets in accordance with (40) provide faithful representation of the group G_c/H of canonical transformations which change physical states.

All of this can be carried through for instantaneous action-at-a-distance equations which satisfy the Currie-Hill conditions. But there is an embarrassment of riches -a great many canonically inequivalent Hamiltonian formulations can be obtained. For example, multiplying all of the U_i by the same constant yields a new Hamiltonian formulation canonically inequivalent to the old. Which canonical formulation is then to be chosen?For a non-dissipative dynamics in which the motion reduces to free particle motion at large interparticle separations, this ambiguity can be resolved (up to canonical transformation) by the demand of asymptotic reduction to the usual free particle Hamiltonian formulation [27]. The argument begins by re-writing the conditions (30) which determine the U_i in the form

$$D U_j + \sum_{i=0}^{2N} U_i (\partial h_i / \partial y_j) = \partial \Gamma / \partial y_j \tag{42}$$

where

$$\Gamma = \sum_{i=0}^{2N} U_i h_i \tag{43}$$

Make the definition

$$\Psi_j^\alpha = L_\alpha U_j + \sum_{i=0}^{2N} U_i (\partial q_i^\alpha / \partial y_j) - \partial \Omega_\alpha / \partial y_j \tag{44}$$

The condition (37) is then just $\Psi_i = 0$. It is straightforward to show that

$$D \Psi_j^\alpha = \partial (D G_\alpha) / \partial y_j - \sum_{i=1}^{2N} \Psi_i^\alpha (\partial h_i / \partial y_j) \tag{45}$$

with D and G_α given by (35) and (39). Now let the U_i , $i \neq 0$ be prescribed on some hypersurface which does not lie along the characteristics of the partial differential operator D (these characteristics are just the solution curves of (27); D is the substantive derivative which effects an infinitesimal transformation along these solution curves), and let Γ be prescribed everywhere (Prescribing Γ removes the arbitrariness associated with adding a gradient to the U_i . Different choices for Γ give rise to canonically equivalent Hamiltonian formulations). Let $\Psi_i^\alpha = 0$ on this hypersurface, and prescribe Ω_α off of the hypersurface by the demand that G_α be a conserved quantity so that $DG_\alpha = 0$. Then (42) propagates the values of the U_i on the hypersurface along the solution curves of (27) [along the characteristics of D] while (45) propagates the condition

$\Psi_j^\alpha = 0$ for the canonical representation of the transformation (31) along the solution curves of (27).

The above analysis is applied to relativistic dynamics by taking the hypersurface to be at infinite interparticle separation where the particles are free. The U_i are given the forms appropriate to the usual free particle Hamiltonian formulation, for which the Poincare group is canonically represented; (42) and (45) then propagate this canonical formulation into the interacting region. If the particles are free at $t = -\infty$ and at $t = +\infty$ but interact at finite times, one is of course not entitled to prescribe initial conditions at both $t = -\infty$ and $t = +\infty$. This, however, is a problem only for dissipative systems.

The relation between the original physical particle coordinates and canonical coordinates is determined by the solution to Pfaff's problem. Only in regions where the particles move like free particles can these be identical; when the particles interact, the canonical coordinate differs from the physical particle coordinate by an interaction piece. The same thing happens to constants of the motion, which also acquire an interaction piece (for the Hamiltonian and the linear momentum, this is required by a zero-interaction theorem which appears in one of the van Dam-Wigner papers[8]). That this should happen is not surprising when one compares the action-at-a-distance and field theoretic descriptions of electrodynamic interactions. If one ignores difficulties with infinite self-interaction, then the Hamiltonian H for the field theoretic description has, in an obvious notation, the form

$$H = H_{\text{particle}} + H_{\text{field}} + \lambda H_{\text{particle-field}} \tag{46}$$

where λ is a coupling constant. One can contemplate pushing the particle-field coupling to higher order in λ via successive canonical transformations; when this is done, the old canonical particle position coordinates, differ from the new canonical coordinates by an interaction piece. It would be of considerable interest to see in detail what relationship, if any, the new particle Hamiltonian obtained via this process has to the particle Hamiltonian obtained from the Newtonian format action-at-a-distance theory via the Lie-Konigs theorem.

The discussion of the approach to a Hamiltonian dynamics via the Lie-Konigs theorem would not be complete without some discussion of the problems and the prospects. The transition from classical Hamiltonian mechanics to quantum mechanics is plagued with ambiguities, such as operator ordering. It may be that one should start with a quantum

theory right away -or that the quantum action-at-a-distance theory should be obtained via a suitable reduction from quantum field theory with the classical action-at-a-distance theory serving only as a useful guide- or that quantization methods developed in the years since Ed Kerner and I worked on these problems will provide the answer. Another difficulty is the messiness of the results for realistic interactions, such as those of electrodynamics, which Kerner and I have always regarded as the touchstone. This messiness suggests that either the element of simplicity in instantaneous action-at-a-distance dynamics of the kind Kerner and I worked with has not been found, or that the action-at-a-distance view is not fundamental, but merely a useful computational tool. On the other hand, the action-at-a-distance viewpoint may provide insights not otherwise easily available. For example: the fact that expansion in powers of e^2 is really expansion in powers of $e^2/(mc^2r)$ at the classical level, where r is an interparticle separation, suggests that the piling-up of factors of $1/r$ in non-relativistic reductions from quantum electrodynamics may arise from ignoring the role played by the classical electron radius $e^2/(mc^2)$.

It is always important to know the limitations of a theoretical point of view. Towards this end I would like now to discuss simple example, drawn from electrodynamics, for which the predictive point of view appears to fail at high energies. The example is the special case of the half-advanced half-retarded two-body problem of classical electrodynamics in which the motion is confined to one dimension because of initial conditions. The equations of motion are

$$m a_1 = \frac{1}{2}\left[1-(v_1/c)^2\right]^{3/2} e\cdot\left[(E_1)_{adv} + (E_1)_{ret}\right] \tag{47a}$$

$$m a_2 = \frac{1}{2}\left[1-(v_2/c)^2\right]^{3/2} e\left[(E_2)_{adv} + (E_2)_{ret}\right] \tag{47b}$$

where

$$(E_1)_{ret} = \left[\frac{e}{x_{12}^2}\cdot\frac{1+v_2/c}{1-v_2/c}\right]_{ret} \tag{48a}$$

$$(E_1)_{adv} = \left[\frac{e}{x_{12}^2}\cdot\frac{1-v_2/c}{1+v_2/c}\right]_{adv} \tag{48b}$$

$$(E_2)_{ret} = \left[\frac{-e}{x_{12}^2}\cdot\frac{1-v_1/c}{1+v_1/c}\right]_{ret} \tag{48c}$$

$$(E_2)_{adv} = \left[\frac{-e}{x_{12}^2} \cdot \frac{1 + v_1/c}{1 - v_1/c} \right]_{adv} \qquad (48d)$$

Here v_i and a_i are the velocity and acceleration of the i^{th} particle, $x_{12} = x_1(t_1) - x_2(t_2)$ is difference between the position of particle 1 at time t_1 and particle 2 at time t_2 . E_i is the electric field felt by the i^{th} particle; subscripts adv , ret indicate whether the positions and velocities of the other particle are to be evaluated at the advanced or retarded time. The particle's charge and mass are denoted by e and m ; c is the velocity of light.

Consider now the case of symmetric motion, for which $v_1(t) + v_2(t) = 0$. This problem has been solved numerically at low to intermediate energies by Anderson and von Baeyer[28], and in the high energy limit by myself. If one plots the distance of closest approach against energy, it looks something like the sketch in figure 4 . The iterative procedure used by Anderson and von Baeyer failed to converge for asymptotic particle velocities greater than about 0.9545 c , corresponding to an energy (for one of the particles) of about 3.353 mc^2; the distance of closest approach decreased monotonically up to that point, and was (x_{12}) min = 0.9077 $e^2/(mc^2)$ at the point where the iteration would no longer converge. A previously unpublished high-energy-limit solution of my own has a distance of closest approach which is exactly $e^2/(mc^2)$. Thus one has a problem for which the Newtonian initial data $v_1 = v_2 = 0$, $x_1 - x_2 = d$, $x_1 + x_2 =$ anything is not sufficient to guarantee a unique solution for d lying between 0.9077 e^2/mc^2 and e^2/mc^2 . On the other hand, a theorem of Driver[9] shows that such initial data <u>is</u> sufficient to guarantee existence and uniqueness of the solution for d sufficiently large.

I will finish my lecture by sketching the way in which my high-energy-limit solution can be obtained. The particle world lines for this solution are sketched in figure 5 . The feature which makes a solution possible is the observation that, in the high-energy-limit segment ABC of the world line of particle 1 and segment A′ B′ C′ of the world line of particle 2 are dominated by interactions along one light cone, while segment CDE of the world line of particle 1 and segment C′ D′ E′ of the world line of particle 2 are dominated by interactions along the other light cone. This can already be seen in the numerical computations of Anderson and von Baeyer: their figures 2 and 3 show that the acceleration curve develops a double peak as the energy increases; one peak is due to the advanced interaction while

the other peak is due to the retarded interaction. The one dimensional two-body problem with interaction along only one light cone can be solved exactly, as shown by Robert Rudd and myself[29]; this exact solution is possible because the equations of motion are differential equations rather than differential-difference equations when the interaction occurs along only one light cone. The exact solution with interaction along the light cone from B to B′ can be used as an approximation to the segments ABC and A′ B′ C′; the exact solution with interaction along the light cone from D to D′ can be used as an approximation to the segments CDE and C′ D′ E′. Matching position and velocity for these approximations at C and at C′ leads to the conclusion that the distance of closest approach is $e^2/(mc^2)$ in the high energy limit. Half of the turning for particle 1 occurs at B , with the other half at D .

Now for the details, which make use of the results for the world lines with interaction along one light cone obtained by Rudd and Hill[29], hereafter refered to as RH . For simplicity the velocity of light c is 1 . The results from RH are used with $m_1 = m_2 = m$ and $e_1 e_2 = e^2/2$ (since we are dealing with half advanced - half retarded interactions). The needed expressions for the world lines are given by RH equations (11)-(15); in the high energy limit these can be adequately approximated by

$\mu = E^2(E^2 - 4m^2)$, $b_1 = b_2 = 2m^3 e^2 E^{-2}(E^2 - 4m^2)^{-1}$, and

$$y_{f1} = x_1 - (K/E) - x_0 \tag{49a}$$

$$y_{f2} = x_2 - (K/E) + x_0 \tag{49b}$$

$$x_0 = \tfrac{1}{2} e^2 (E^2 - 2m^2) \cdot E^{-1} (E^2 - 4m^2)^{-1} \tag{50}$$

$$t_1 - t_{10} = \theta_1 \cdot E\,(y_1^2 - b_1^2)^{1/2} \cdot (E^2 - 4m^2)^{-1/2} \tag{51a}$$

$$t_2 - t_{20} = \theta_2 \cdot E\,(y_2^2 - b_2^2)^{1/2} \cdot (E^2 - 4m^2)^{3/2} \tag{51b}$$

$$t_{10} - t_{20} = 2\theta x_0 \tag{52}$$

The $\tanh^{-1}$ terms in RH equations (13)-(15) have been neglected in obtaining the approximations (49)-(52). These $\tanh^{-1}$ terms, which are down by a factor $(m/E)^5$ in RH eqs. (13) and (14), and by a factor $(m/E)^4$ in RH eq. (15), contain the $\ln t$ dependence which is characteristic of coulomb interactions in the asymptotic region, but with the

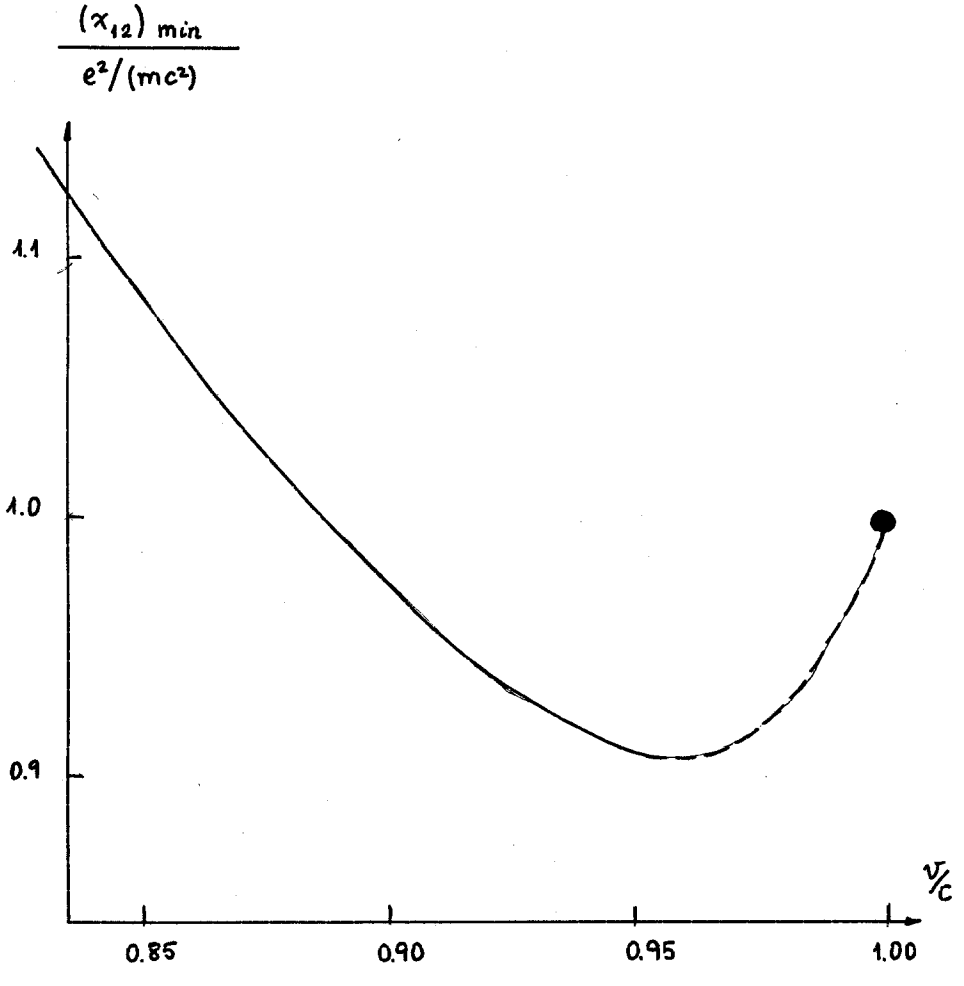

Figure 4. Distance of closest approach versus asymptotic velocity.

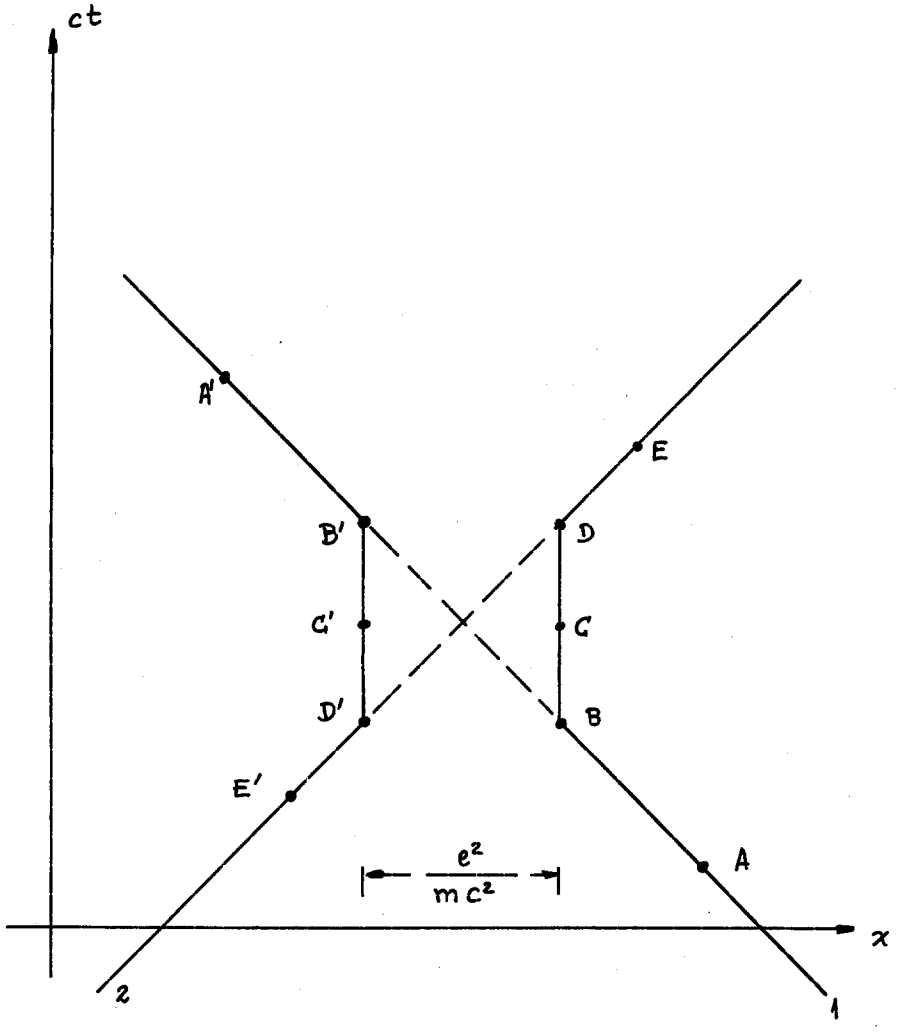

Figure 5. The half advanced half retarded problem in the high energy limit.

wrong coefficient since interactions along only one of the light cones are taken into account.

In order that the world lines be symmetrically placed with respect to the origin $x = t = 0$, demand that the value of x_1 and x_2 at their respective turning points $x_{10} = (K/E) + x_0 + b_1$ and $x_{20} = (K/E) - x_0 - b_2$ be the negatives of one another (this implies $K = 0$) and that the turning times t_{10} and t_{20} be negatives of one another; this implies

$$t_{10} = \theta x_0 \quad \text{and} \quad t_{20} = -\theta x_0$$

To obtain an approximation to the world line of the half-advanced half retarded problem for $t_1 < 0$ and $t_2 > 0$, choose $\theta = -1$ and make the Lorentz transformation

$$x = \frac{x' + vt'}{\sqrt{1 - v^2}} \quad ; \quad t = \frac{t' + vx'}{\sqrt{1 - v^2}}$$

where $v = E^{-1}(E^2 - 4m^2)^{1/2}$. The transformed world line of particle 1 (wich gives a high-energy approximation for the first part of the motion) written in terms of v (instead of E) is

$$\left(\frac{t_1' + vx_1'}{\sqrt{1 - v^2}} + x_0\right)^2 = \frac{1}{v^2}\left[\left(\frac{x_1' + vt_1'}{\sqrt{1 - v^2}} - x_0\right)^2 - b_1^2\right]$$

This can be rewritten in the form

$$\left[x_1' - x_0\sqrt{\frac{1+v}{1-v}}\right]\cdot\left[x_1' + \frac{2v}{1+v^2}t_1' - x_0\frac{(1-v)\sqrt{1-v^2}}{1+v^2}\right] = \frac{b_1^2}{1+v^2} \qquad (53)$$

which can easily be shown to be the equation of a hyperbola. The asymptotes are

$$x_1' - x_0\sqrt{\frac{1+v}{1-v}} = 0 \qquad (54a)$$

$$x_1' + \frac{2v}{1+v^2}t_1' - x_0\frac{(1-v)\sqrt{1-v^2}}{1+v^2} = 0 \qquad (54b)$$

In the high energy limit these simplify to

$$x_1' \cong \frac{e^2}{2m} \quad , \quad x_1' \cong -t_1'$$

This is the origin of the claim that the distance of closest approach in the high energy limit is just the classical charge radius e^2/m. One can also see that the scale of length and time in the region where the velocity changes rapidly is set by $b_1 \cong 2(e^2/m)(m/E)^4$ in the high energy limit.

The other half of particle one's world line, as well as the $t<0$ and $t>0$ pieces of particle two's world line can be approximated with similar hyperbolas. It is also possible to approximate the whole world line by the somewhat more complicated analytic form

$$\left[x_1' - x_0\sqrt{\frac{1+v}{1-v}}\right]\cdot\left[x_1' + \frac{2v}{1+v^2}t_1' - x_0\frac{(1-v)\sqrt{1-v^2}}{1+v^2}\right]\cdot\left[x_1' - \frac{2v}{1+v^2}t_1' - x_0\frac{(1-v)\sqrt{1-v^2}}{1+v^2}\right] = \frac{2b_1^2}{1+v^2}\left[x_1' - x_0\frac{(1-v)\sqrt{1-v^2}}{1+v^2}\right] \tag{55}$$

which, in the high-energy limit, reduces to each of the two approximating hyperbolas in each of the two regions. This can be seen by writing it in the form

$$\left[x_1' - x_0\sqrt{\frac{1+v}{1-v}}\right]\cdot\left[x_1' + \frac{2v}{1+v^2}t_1' - x_0\frac{(1-v)\sqrt{1-v^2}}{1+v^2}\right]$$

$$= \frac{b_1^2}{1+v^2}\cdot\left\{1 + \frac{\left[x_1' + \frac{2v}{1+v^2}t_1' - x_0\frac{(1-v)\sqrt{1-v^2}}{1+v^2}\right]}{\left[x_1' - \frac{2v}{1+v^2}t_1' - x_0\frac{(1-v)\sqrt{1-v^2}}{1+v^2}\right]}\right\} \tag{56}$$

The second (messy) term in the curly bracket on the right hand side is very small in the high-energy-limit during the $t_1'<0$ rapid velocity change. This approximate form has the advantage that it interpolates smoothly between the two hyperbolas in the high-energy limit, with $dx_1'/dt_1' = 0$ when $t_1' = 0$ as should be the case.

It appears to be possible to use these insights into the high-energy behavior as the basis for a numerical exploration of the region between the point where the Anderson-von Baeyer numerical scheme broke

down and the high energy limit. I hope that someone will take up the challenge and do this. It should perhaps be remarked that this unexpected appearance of a minimum distance of closest approach appears to be limited to the half advanced half retarded problem; numerical calculations by Huschilt and Baylis[30] show no evidence of this for the case of purely retarded interactions with radiation damping.

In closing I would like to acknowledge numerous discussions of these matter with Ed Kerner over the years. I would also like to thank the Conference organizers, F. Fayos, X. Fustero, J. Gomis, V. Iranzo, J. Llosa, J.A. Lobo, F. Marqués, A. Molina, A. Poch, J.M. Pons and J. Porta for their many hours of work which made this conference possible.

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SINGULAR LAGRANGIAN FORMALISM IN PARTICLE DYNAMICS, I .

G. Longhi

Istituto di Fisica Teorica dell'Università
Firenze

Istituto Nazionale di Fisica Nucleare
Sezione di Firenze

The problem of singular Lagrangians was studied since the time of Weierstrass, in the context of the calculus of variations (the so-called homogeneous case, see for instance H. Rund[1]). In the more recent literature, one of the first physicist who studied the problem was perhaps Dirac[2] .

Perhaps the first recongnised physical example where such a problem appears is that of the electromagnetic field, considered as a dynamical system with an infinite number of degrees of freedom, where the canonical momenta conjugated to the scalar potential vanish, so providing a first example of a set of canonical constraints.

Another simple example is the Lagrangian of a scalar massive relativistic particle, written in a manfiest invariant form[3] , of which a great number of models are natural generalizations[4] .

A number of papers could be quoted, in which this problem is in one or in another way discussed, anyway its systematic study from the point of view of the physical applications and of the quantum theory is due again to Dirac, who developed it in the well known series of papers, and subsequently reorganized in the lessons at the Yeshiva University[5] .

Let me show in a simple way how the interest in singular lagrangians arises in particle mechanics. The action for a single material point is

$$S = -\int_{x'}^{x''} ds \quad , \qquad ds = \sqrt{dx^\mu dx_\mu} \tag{1}$$

(signature (+, ---)),
x^μ being the set of lagrangian coordinates which determines the

event in Minkowski space at which the point is observed. It is supposed that the observation is made in an inertial reference frame, and $x^0 = t$ ($c = 1$) is the time measured in this frame. In (1) x' and x'' are given events and the integral is on the possible paths from x' to x''.

A possible generalization to two interacting points is[6,7]

$$S = -\int_{x_1', x_2'}^{x_1'', x_2''} \left[\sqrt{u_1(r^2)\, dx_1^2} + \sqrt{u_2(r^2) dx_2^2}\right] \tag{2}$$

where $r^\mu = x_1^\mu - x_2^\mu$, and the integral is extended to paths from (x_1', x_2') to (x_1'', x_2'') in the configuration space (x_1, x_2).

From these examples we may say that we are interested in the study of actions of the form

$$S = \int_{x_i'}^{x_i''} L(x_i, dx_i) \qquad i = 1, \ldots, n \tag{3}$$

where L is homogeneous of first degree in the second argument.

A problem of this kind could admit a geometrical interpretation, namely the problem to find geodesic in a space where the distance is defined by

$$ds = L(x, dx) \tag{4}$$

This would define a Finsler space[1] if some conditions on L were met: homogeneity of first degree in dx, positivity and non singularity of the matrix tensor g_{ij} defined as

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial \dot{x}_i \partial \dot{x}_j} \tag{5}$$

where a parameter τ was introduced for convenience:

$$\dot{x}_i = \frac{d}{d\tau} x_i \tag{6}$$

Now in the example given in (1) g is indeed not singular, but in the case of the example given in (2) g is singular. This means that in general equation (4) doesn't define a Finsler space as usually defined (see especially the second reference quoted in[1]).

The choice of a parameter τ (the paths on which the integral in equation (3) is performed are lines in total configuration space) is useful, but not strictly necessary. If this is done we may define a singular Lagrangian $L(x, \dot{x})$:

$$S = \int_{\tau'}^{\tau''} L(x, \dot{x})\, d\tau \tag{7}$$

where $x_i' = x_i(\tau')$, $x_i'' = x_i(\tau'')$. L will be homogeneous of first degree in $\dot{x}_i$. Clearly we may change at will the parameter τ . The action doesn't depend on τ ; it is a functional of $x_i(\tau)$ and a function of x_i' and x_i'' .

The homogeneity of L implies that the canonical hamiltonian vanishes:

$$H_c = p_i \dot{x}_i - L = \frac{\partial L}{\partial \dot{x}_i} \dot{x}_i - L \equiv 0 \tag{8}$$

and by differentiating on $\dot{x}_j$:

$$\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} \dot{x}_i = 0 \tag{9}$$

that is the hessian matrix $\left\| \frac{\partial^2 L}{\partial \dot{x}_j \partial \dot{x}_i} \right\|$ has at least one eigenvector corresponding to a null eigenvalue. This shows that a parameter invariant lagrangian is a particular case of a degenerate lagrangian, where for a degenerate lagrangian it is understood a lagrangian such that

$$\left| \frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} \right| = 0 \tag{10}$$

It is worthwhile to observe that any lagrangian can be promoted to a singular lagrangian in a space of coordinates with one more dimension (for a general statement see again Rund[1]).

Degenerate lagrangians are discussed in detail for instance by Shanmugadhasan[8] and by Sudarshan and Mukunda[9] .

The property (10) has a lot of deep consequences. First of all on the equations of motion

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \tag{11}$$

which can be written

$$\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} \ddot{x}_j = \frac{\partial L}{\partial x_i} - \frac{\partial^2 L}{\partial x_j \partial \dot{x}_i} \dot{x}_j \tag{12}$$

by performing explicitly the derivative, and by taking into account the independence on τ of L ; the r.h.s. of equations (12) is a function of x_i and $\dot{x}_i$, whereas the "accelerations" $\ddot{x}_i$ are present

only in the l.h.s., where they appear explicitly. Now, due to (10), only some of the $\ddot{x}_i$ can be calculated from (12). By linearly combining the n equations (12) we can find (at most) n-r relations involving x_i and $\dot{x}_i$ only, where r is the rank of the Hessian matrix:

$$B_\alpha (x_i, \dot{x}_i) = 0 \quad , \qquad \alpha = 1, 2, \ldots \leq n-r \tag{13}$$

These are called lagrangian constraints, they put restrictions on the initial conditions.

By differentiating in τ these equations, and using again the equations of motion (12), we can eventually find other relations of the kind of the B_α , and so on. This procedure can be found in all details in references 8) and 9) . It turns out at the end that a certain number p of canonical constraints may appear

$$\Omega_\rho (p_i, x_i) = 0 \quad , \qquad \rho = 1, \ldots, p \tag{14}$$

where the momenta p_i are defined in the usual way

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \tag{15}$$

(but due to eq. (10) these cannot be inverted in terms of the $\dot{x}_i$), and some of the "velocities" $\dot{x}_i$ will remain undetermined. Among the constraints (14) are present the so called primary constraints, which follow simply from the definition of the p_i , equation (15), by eliminating between these equations the $\dot{x}_i$, these are identities in the space $(x_i, \dot{x}_i)$ and are in number n-r . It will be in general $p \geq n - r$.

The constraints $\Omega_\rho (p,x) = 0$ can be classified in constraints of first and second class according to Dirac[5] . First class constraints are such that their Poisson brackets ($\{x_i, p_j\} = \delta_{ij}$) with all the other constraints are zero as a consequence of the vanishing of the $\Omega_\rho (p,x)$ (of all the Ω_ρ in general), or, as is usually referred to, they have weakly vanishing Poisson brackets with the other constraints. Otherwise they are second class constraints.

Usually in particle mechanics second class constraints appear (for more than 1 particle of course), but in field theory the situation can be different: the relativistic string model and the electromagnetic field are two examples of models with an infinite number of first class constraints.

In the case of particle mechanics the situation as seen from the lagrangian point of view seems to be very different from that of Droz-

Vincent[10] Todorov[11] and Komar[12], based on a set of postulated first class constraints. Nevertheless, as we will see on a worked example, there is a sort of correspondence.

The sequel of this lecture will be devoted to the analysis of the correspondence and to the advantages (and disadvantages) of the lagrangian approach.

To make things as clear as possible I find it convenient to discuss one simple and workable model, namely the two-body lagrangian model firstly proposed by Kamimura and Shimizu[6] and later discussed by Dominici et al[7] . This model has the action given by equation (2), where the potentials U_i (i=1,2) are given by

$$U_1 = U_2 = m^2 - V(r^2) \tag{16}$$

(we will consider the case of equal masses for simplicity).

The parametric lagrangian is

$$L(x,\dot{x}) = -\sqrt{m^2 - V(r^2)}\left(\sqrt{\dot{x}_1^2} + \sqrt{\dot{x}_2^2}\right) \tag{17}$$

where $x_i^\mu = x_i^\mu(\tau)$ are the lagrangian coordinates of the two constituents, $i = 1,2$, which are supposed to specify the events of the two particles in a given inertial frame.

By defining the conjugated momenta as

$$p_i^\mu = -\frac{\partial L}{\partial \dot{x}_{i\mu}} \tag{18}$$

where the minus sign is due to the metric signature, we get two primary constraints

$$\left.\begin{aligned} \Omega_1 &= p_1^2 - m_1^2 + V(r^2) \\ \Omega_2 &= p_2^2 - m_2^2 + V(r^2) \end{aligned}\right\} \tag{19}$$

and a secondary constraint :

$$\chi = V'(r^2)\cdot(p,r) \tag{20}$$

where

$$p^\mu = p_1^\mu + p_2^\mu \tag{21}$$

is the total momentum of the system.

It is shown in ref. 7) that, apart from a reparametrization invariance in τ , the constraints (19) and (20) determine a family of covariant word-lines, parametrized by twelve initial data specified at equal τ ; this means that the initial data must be given with a specified time delay.

On the other hand the Droz-Vincent-Todorov-Komar approach to the same problem[10,11,12] would start with two first class constraints, which are the following

$$\left.\begin{aligned} K_1 &= p_1^2 - m_1^2 + V(\underset{\sim}{r}^2) \\ K_2 &= p_2^2 - m_2^2 + V(\underset{\sim}{r}^2) \end{aligned}\right\} \tag{22}$$

where

$$\underset{\sim}{r}^\mu = r^\mu - \frac{(r,p)\,p^\mu}{p^2} \tag{23}$$

which have the property

$$\{K_1, K_2\} = 0 \tag{24}$$

It is known that the model (22) doesn't specify a unique set of world-lines[12,13]. In order to have that we need the addition of one gauge-fixing constraint [14] , which can as well be the constraint (20) (the constraint which fixes the parameter τ should be eventually added in both models). K_1 and K_2 by themselves only specify a set of world-sheets.

We must observe that here we are identifying the phase-space coordinates of the model (22) with those of the lagrangian model. This is necessary in order to do a comparison. In this respect we must not forget that the configuration space of the lagrangian model was identified as the physical one. A different situation appears in the approach of Droz-Vincent[15] , where the identification of the physical coordinates is made a posteriori.

We may verify explicitly that the constraints K_1 and K_2 do not specify world-lines, by the use of the explicit form of the Hamilton-Jacobi's function, which in the case of model (22) is known to exist, due to the property (24).

I will give here only the result for the case of harmonic potential $V(r^2) = cr^2$; the calculation of the Hamilton-Jacobi's function

follows the classical procedure based on integrable systems, which will be discussed in the third lecture.

The H-J function $S = S(x,r)$, defined by

$$dS = - p^\mu dx_\mu - q^\nu dr_\nu \tag{25}$$

where the collective canonical coordinates p,x,q and r are defined as:

$$\left.\begin{aligned} x &= \tfrac{1}{2}(x_1+x_2) \quad , \quad & r &= x_2 - x_1 \\ p &= p_1 + p_2 \quad , \quad & q &= \tfrac{1}{2}(p_2 - p_1) \end{aligned}\right\} \tag{26}$$

is given, for the model (22) with $V(r^2) = cr^2$, by

$$S = -(k\cdot x) + \frac{1}{2\sqrt{c}} \sum_{\lambda=1}^{3} \left[\sqrt{c}\, \underline{r}_\lambda \sqrt{\epsilon_\lambda - c\, \underline{r}_\lambda^2} + \right. \\ \left. + \epsilon_\lambda \cdot \arcsin\left(\sqrt{c}\, \underline{r}_\lambda / \sqrt{\epsilon_\lambda}\right)\right] \tag{27}$$

where

$$\underline{r}_\lambda = \epsilon_\lambda^\mu(p)\, r_\mu \quad , \quad \lambda = 1,2,3 \tag{28}$$

The $\epsilon_\lambda^\mu(p)$ being the polarization four vectors for a massive particle defined by Weinberg[16], and claculated in the second paper of ref. 7). In eq. (27) k^μ is such that

$$k^2 = M^2 = 4m^2 + \sum_\lambda \epsilon_\lambda \tag{29}$$

and $\vec{k}$, ϵ_λ are six arbitrary constants of integration.

It can be verified that S is solution (a complete integral of the two equations

$$\left.\begin{aligned} p^2 + 4q^2 + 4m^2 - 4c\, \underset{\sim}{r}^2 &= 0 \\ (pq) &= 0 \end{aligned}\right\} \tag{30}$$

that is of equations $K_1 = 0$, $K_2 = 0$ expressed in terms of the collective coordinates (26), if these equations are thought as partial differential equations (Hamilton-Jacobi's equations) in S through

$$p^\mu = - \frac{\partial S}{\partial x_\mu} \quad , \quad q^\nu = - \frac{\partial S}{\partial r_\nu} \tag{31}$$

The solutions of the equations of motion can be got from S as usual, by differentiating S with respect to the six constants ε_λ and $\vec{k}$, and by putting the results equal to new six constants.

The solution so obtained is in a non parametric form, suitable for our discussion:

$$\frac{\partial S}{\partial k_i} = h_i = -\frac{k_i}{k_o} x_o + x_i + \sum_\lambda \sqrt{\varepsilon_\lambda - c r_\lambda^2} \cdot \frac{\partial r_\lambda}{\partial k_i} \tag{32}$$

where

$$\frac{\partial r_\lambda}{\partial k_i} = \frac{k_\lambda r_i - \delta_{\lambda i} (\vec{k}\cdot\vec{r})}{M(k_o + M)} - \frac{\bar{r}}{M^2}\left(\delta_{\lambda i} - \frac{k_\lambda k_i}{k_o (k_o + M)}\right) \tag{33}$$

and where

$$\bar{r} = (k \cdot r) \tag{34}$$

$$k_o = \sqrt{\vec{k}^2 + M^2} = \sqrt{\vec{k}^2 + 4m^2 + 4\sum_\lambda \varepsilon_\lambda} \tag{35}$$

The other set of solutions is

$$\frac{\partial S}{\partial \varepsilon_i} = a_i = -\frac{2}{k_o} x_o + \frac{2}{M^2} \sum_\lambda \frac{k_\lambda \bar{r}}{M k_o} \sqrt{\varepsilon_\lambda - c r_\lambda^2} + \frac{1}{2\sqrt{c}} \arcsin \frac{\sqrt{c}\, r_i}{\sqrt{\varepsilon_i}} \tag{36}$$

The two sets of equations (32) and (36) should be solved in terms of say $\vec{x}$ and $\vec{r}$, so giving two solutions of the form

$$\left.\begin{aligned} r_i &= g_i (x_o, r_o; k_i, \varepsilon_i) \\ x_i &= f_i (x_o, r_o; k_i, \varepsilon_i) \end{aligned}\right\} \qquad (i=1,2,3)\ , \tag{37}$$

which represent a 2-dimensional surface, for any given set of the 12 integration constants k_i, ε_i, in the configuration space.

It can be verified that it doesn't happen that $\vec{x}_1$ and $\vec{x}_2$ are function of x_1^o and x_2^o separately; it follows that this surface is not the product of two curves or world-lines in the two Minkowski spaces M1 and M2 of the two particles, but when projected on the two subspaces M1 and M2 it gives world-sheets.

It is not easy to verify this fact explicitely, since it is not possible to solve in closed form the equations (32) and (36) in $\vec{x}_1$ and $\vec{x}_2$. But it would be sufficient to find a particular case where it happens for the effectiveness of the general statement. A particular case is the choice of the rest frame of p^μ, $\vec{p} = 0$. Since from the equations (32) and (36) and (31) we get p^μ = constants = k^μ, we have

to choose $\vec{k} = 0$. Now the equations (32) and (36) are easily solved in $\vec{x}_i$, (i=1,2) ; we get

$$\left.\begin{aligned} x_{1i} &= h_i + \frac{\sqrt{\epsilon_i}}{M}(t_2-t_1)\cos\phi_i - \frac{1}{2}\sqrt{\frac{\epsilon_i}{c}}\sin\phi_i \\ x_{2i} &= h_i + \frac{\sqrt{\epsilon_i}}{M}(t_2-t_1)\cos\phi_i + \frac{1}{2}\sqrt{\frac{\epsilon_i}{c}}\sin\phi_i \end{aligned}\right\} \qquad (38)$$

where

$$\phi_i = \ell\sqrt{c}\left(a_i + \frac{t_1+t_2}{M}\right) \qquad (39)$$

and

$$t_1 = x_1^0 \quad , \quad t_2 = x_2^0$$

The conclusion is that this model doesn't give world-lines, at least until we mantain the identification of the configuration space with the physical space.

Another possibility is left open, that is to look for new physical coordinates such that the surface defined by $k_i = 0$, corresponding to some choice of the integration constants, will appear to be the product of two curves in the new physical subspace M_1 and M_2 . This different interpretation would determine the Droz-Vincent physical positions[15] .

Inasmuch we have chosen the physical coordinates as I did, the only way to have world-lines instead of world-sheets is to put a restriction (gauge-fixing constraints) on the solutions (37). If we choose such a restriction to be

$$\bar{r} = (k\cdot r) = 0 \qquad (40)$$

that is the same constraint given as secondary constraint by the lagrangian model, we get exactly the same solutions we had got by starting from the (19) plus (20) (with $V' \neq 0$). That is, putting $\bar{r} = 0$ in the equations (32) and (36) , we get

$$h_i = -\frac{k_i}{k_0}x_0 + x_i + \sum_\lambda \sqrt{\epsilon_\lambda - c\underline{r}_\lambda^2}\cdot\frac{k_\lambda \underline{r}_i - \delta_{\lambda i}(\vec{k}\cdot\vec{r})}{M(k_0+M)} \qquad (41)$$

$$a_i = -\frac{2}{k_0}x_0 + \frac{1}{2\sqrt{c}}\,\text{arc sin}\,\frac{\sqrt{c}\,\underline{r}_i}{\sqrt{\epsilon_i}} \qquad (42)$$

together with

$$r_o = \frac{\vec{k}\cdot\vec{r}}{k_o} = \frac{\vec{k}\cdot\vec{\xi}}{k_o} \tag{43}$$

These can be easily solved in $\vec{r}$ and $\vec{x}$, giving

$$\begin{aligned} x_i &= \bar{h}_i + \frac{k_i}{k_o} x_o \\ r_i &= \sum_\lambda \left[\delta_{i\lambda} + \frac{k_i k_\lambda}{M(k_o+M)}\right]\cdot\sqrt{\frac{\varepsilon_\lambda}{c}}\cdot \sin\left[2\sqrt{c}\left(a_\lambda + \frac{2x_o}{k_o}\right)\right] \\ r_o &= \frac{1}{M}\sum_\lambda k_\lambda\sqrt{\frac{\varepsilon_\lambda}{c}}\cdot \sin\left[2\sqrt{c}\left(a_\lambda + \frac{2x_o}{k_o}\right)\right] \end{aligned} \tag{44}$$

where $\bar{h}_i$ is the new constant:

$$\bar{h}_i = h_i - \sum_\lambda \sqrt{\frac{\varepsilon_i}{c}}\, k_\lambda\cdot\sqrt{\varepsilon_\lambda}\cdot \sin\left[2\sqrt{c}\,(a_i - a_\lambda)\right] \tag{45}$$

The solution (44) depends on x_o only, besides the twelve constants ε_i, k_i, $\bar{h}_i$, a_i, and a time correlation appeared between r_o and x_o, that is between t_1 and t_2. This is exactly the solution which could be got from the lagrangian equations of motion, once we had eliminated the parameter τ in terms of x_o. The situation is sketched in Fig. 1, where the two world-sheets are indicated with σ_1 and σ_2 and the two world lines, determined with the condition $\bar{r}=0$, with γ_1 and γ_2.

We could at this point reconstruct a 2-dimensional surface in the total configuration space as the product of γ_1 and γ_2. On this new surface the condtion $\bar{r} = (p.r) = 0$ will hold only on the line γ, of which γ_1 and γ_2 are the projections on M_1 and M_2 (see Fig. 1). Any line on this surface will give the same world-lines γ_1 and γ_2, and since it is a 2-dimensional surface, points on it can be parametrized with two independent parameters τ_1 and τ_2, which can be used to give an independent parametrization for the two world-lines.

From this point of view we can make contact with the predictive point of view[17]. Indeed from the world-lines so obtained, which are now parametrized at will, we may look for forces which are defined to act at the same time in the chosen frame (and in any reference frame). A procedure which is possible in principle is the following: we may take the equations defining the two world-lines at the same t, by eliminating τ_1 and τ_2 in terms of t, and differentiate these twice in t. We can eliminate the integration constants in order to get the instantaneons forces. This is a classical argument in favor of

the existence of action-at-a-distance forces.

Let me now summarize the situation as comes out from the analysis of this particular model. The Todorov-Komar approach[11,12], when applied to the two-body problem, gives the same results as the lagrangian model, if we select between all possible gauge-fixing constraints (those not τ dependent) the same constraint which arises from the lagrangian equations of motion. Alternatively we can take the general solution of the two equations $K_i = 0$ (understood as Hamilton-Jacobi's equation), and look for new physical coordinates, in terms of which the 2-dimensional integral surface become product of world-lines.

In any case it results that the lagrangian approach, starting from a set of physical coordinates, gives a complete dynamical scheme, in the sense that it specifies not only the world-lines, but also the time correlation which is a necessary element of the law of force.

At this point we can try to list some of the advantages and some of the disadvantages of the lagrangian approach.

One of the first advantages is that the requirements of relativistic invariance, and of other kind of invariance, is more easily carried out in the lagrangian approach. Even the requirement of separability (or cluster decomposition) seems to be more clearly accomplished. Moreover, an interaction with an external field is more easily introduced in a lagrangian, where it can be more clear the way it can be coupled to the particle coordinates, in order to preserve some symmetry.

On the other hand it can be very difficult to find a lagrangian with a given set of primary and secondary constraints. Given the primary constraints it can be impossible, due to algebraic difficulties, to determine the corresponding lagrangian. And more, there will be ambiguities in the choice of a lagrangian, since the classification of a set of constraints in primary and secondary ones is arbitrary, and it is of small if not null physical meaning[5].

As an illustration of this situation let me again take the example of the two-body system. I know three lagrangians which give rise to the set of constraints (in the case of equal masses):

$$p^2 + 4q^2 + U(r^2) = 0 \tag{46}$$

$$(pq) = 0 \tag{47}$$

$$(p\,r) = 0 \tag{48}$$

The first in chronological order is that proposed by Kalb and Van Alstine[18] and by Takabayasi[19]

$$L = -\sqrt{-U(r^2)\left[\dot{x}^2 - \frac{(\dot{x}r)^2}{r^2} + \beta^2 r^2\right]} \tag{49}$$

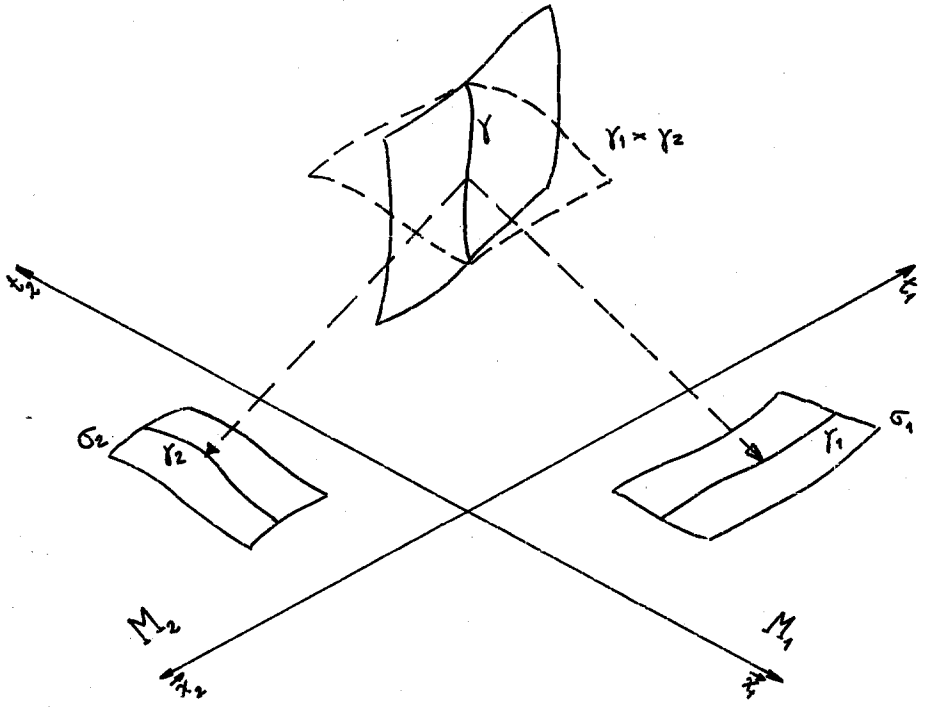
which gives the constraints (46) and (48) as primary and (47) as secondary constraint.

The second is that proposed by Kamimura and Shimizu[6], equation (17), which gives the constraints (46) and (47) as primary and (48) as secondary constraint.

Finally a lagrangian proposed by Gomis, Lobo and Pons[20], given by

$$L = -\frac{1}{2}\sqrt{-U(r^2)}\cdot\left[\left(\sqrt{\dot{x}_1^2} + \sqrt{\dot{x}_2^2}\right) + \frac{(\dot{r}r)^2}{r^2}\right]^{1/2} \tag{50}$$

gives all the constraints (46), (47) and (48) as primary constraints. In the next lecture we will see what the lagrangian approach can suggest on the N-body problem, where the main difficulty is the realization of the separability or cluster decomposition property.



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SINGULAR LAGRANGIAN FORMALISM IN PARTICLE DYNAMICS, II.

G. Longhi
Istituto di Fisica Teorica, Univ. di Firenze
Sezione I.N.F.N. di Firenze

with

D. Dominici
Sezione I.N.F.N. di Firenze

and

J. Gomis and J.A. Lobo
Departamento de Física Teórica
Universidad de Barcelona

In this lecture I will give a short account of the various approaches to the N-body problem ($N > 2$), and I will discuss the problem posed by the requirement of separability (I will consider the spinless case only).

The requirement of separability, or cluster decomposition property, is a well known difficulty encountered in the study of N-bodies in direct interaction. The necessity of requiring it was first raised on by Foldy[1], and investigated by several authors[2].

Several model satisfying this requirement have been studied, starting from different points of view. I will not give here a complete review; recent reviews on this subject can be found in references 3)4) and 5).

Separability, or cluster decomposition property, will be here understood in the following meaning: when a system of N interacting particles breaks into two or more dynamically independent clusters, because of a finite range character of the interactions, or because there is a large space-like distance between them and the mutual interactions vanish in this limit, the set of constraints must as well break into

two or more dynamically independent clusters, because of a finite range character of the interactions, or because there is a large space-like distance between them and the mutual interactions vanish in this limit, the set of constraints must as well break into two or more corresponding subgroups, each subgroup describing the separate dynamics of each cluster of particles.

An important consequence of the requirement of separability is the necessity of many-body forces in the direct interaction dynamics of many-bodies[2] . This was shown particularly in references 6), 7) and 8) .

I will give here a short account of some approach to the problem. A first group of models starts with the search of N first class constraints with the cluster decomposition property. This kind of approach has been initiated by Todorov[3,9] , Komar[10] and Droz-Vincent[11]. Generally speaking, in this kind of approach it is neccessary to specify a set of gauge-fixing constraints (N-1) in order to have a definite dynamics (see lecture Ith. for a discussion of this point, and references quoted therein).

A different approach with both Ith and IIth class constraints has been followed by Gomis et al.[12] , where a constraint of transversality of all the distances between the particles with respect to the total conserved momentum exists, which guarantees the space-like character of the interparticle distances. Due to these constraints the model is not separable according to our definition. Nevertheless the model is predictive[13] , and it should allow a reinterpretation in terms of instantaneous forces. From this point of view the problem of separability disappears, since all that is required is that the world-lines will become straight-lines for large space-like distance, and this is what happens in the model. It exists however the difficulty to attribute a well defined rest mass to each separated free particle.

Other models with a universal potential, which satisfy particular kinds of separability were proposed[14] and[5] . In this last work a model is proposed which is separable in the case of finite range forces.

In this talk I want to discuss a different model, based on both Ith and IIth class constraints, which comes as a suggestion from a lagrangian, which is a natural generalization of the lagrangian for two-bodies proposed in reference 15), 16).

As we have said the Todorov-Komar approach postulates a set of N first class constraints. This set of constraints is supplemented by a set of N gauge constraints, which are necessary in order to eliminate the N temporal coordinates.

On the other hand a singular lagrangian should specify $2N-1$ constraints, that is N mass constraints and the $N-1$ time correlations necessary in order to specify the dynamics (that is the times at which the forces must be calculated), but not the constraint which specify the parameter τ (since the lagrangian itself must be homogeneous of the first degree in the $\dot{x}_i$, that is parameter invariant).

To study the N-body problem I will start again from the lagrangian formulation of the two-body problem[15)16)] :

$$L = -\sqrt{[m^2 - V(r^2)]\,\dot{x}_1^2} - \sqrt{[m^2 - V(r^2)]\,\dot{x}_2^2} \qquad (1)$$

(signature : +,---)
where for simplicity I have chosen equal masses and a unique form of the potential V ; $r^\mu = x_1^\mu - x_2^\mu$.

This parameter invariant lagrangian gives rise to two mass constraints (which are primary constraints, that is identities at the lagrangian level):

$$\left.\begin{aligned} \Omega_1 &= p_1^2 - m_1^2 + V(r^2) \\ \Omega_2 &= p_2^2 - m_2^2 + V(r^2) \end{aligned}\right\} \qquad (2)$$

(Sazdjian[8] has shown that there exists a canonical transformation, by which we may get one and the same potential $V = V_{12} = V_{21}$, starting from two mass constraints with different potentials, $V_{12} \neq V_{21}$, but this last potential would depend in a complicated way on the momenta, or conversely if they depend on r^2 only, the potential $V = V_{12} = V_{21}$ would depend on p_i). Besides, the lagrangian equations of motion imply two lagrangian constraints, one of which is a canonical constraint

$$V'(r^2).\,(p_1 + p_2, r) = 0 \qquad (3)$$

the other being a relation between two undetermined velocities.

Thus this lagrangian selects in a natural way a particular "dynamical" model between the class of models described by two mass constraints of the kind of Todorov-Komar; in this case, two first class constraints are easily recovered from (2), by substituting $\underset{\sim}{r}$ (the transverse part to $p_1 + p_2$ of r) in place of r in the potential V . This will give two other mass constraints equivalent to that given in eq. (2), taking into account eq. (3), when $V' \neq 0$.

All I have said require an important specification. That is what

is the physical space for the coordinates. It was tacitly assumed from the beginning that the space of the lagrangian coordinates is the physical space, in a given inertial frame. Summarizing the situation regarding the two-body problem, we see that the set of three second class constraints (from which we may always select one first class and two second class constraints) given by equation (2) and (3) is equivalent to two Todorov-Komar mass constraints and a gauge-fixing constraint which selects the dynamics and which is separable.

This is a general feature of the singular lagrangian approach. I will not discuss in more details all the problems concerning the gauge-fixing constraints and the world-line conditions, since this was discussed in detail by several authors, and it is the content of a talk by Lusanna at this congress.

The study of the two-body problem suggest that we may try to generalize the lagrangian (1) to N-bodies. Let me start with the three-body case.

A possible generalization of (1) to the three-body case is the following[17]

$$L = -\sum_{i=1}^{3} \sqrt{u_i \cdot \dot{x}_i^2} \tag{4}$$

where we will do the simplifying assumption of equal masses, with

$$\left.\begin{aligned} u_1 &= m^2 - V_{12}(r_{12}^2) - V_{13}(r_{13}^2) \\ u_2 &= m^2 - V_{21}(r_{21}^2) - V_{23}(r_{23}^2) \\ u_3 &= m^2 - V_{31}(r_{31}^2) - V_{32}(r_{32}^2) \end{aligned}\right\} \tag{5}$$

where

$$r_{ij}^{\mu} = x_i^{\mu} - x_j^{\mu} \tag{6}$$

and where we have chosen a simple form of the potentials. We assume also

$$V_{ij} = V_{ji} \tag{7}$$

The primary constraints that follow from (4) and (5) are

$$\Omega_1 = p_1^2 - m^2 + V_{12} + V_{13}$$

$$\Omega_2 = p_2^2 - m^2 + V_{21} + V_{23} \qquad (8)$$
$$\Omega_3 = p_3^2 - m^2 + V_{31} + V_{32}$$

If we analyse the equations of motion given by this lagrangian, we find that two lagrangian constraints must hold

$$\nu_1 \cdot V'_{23}(r_{23}^2) \cdot (p_2 + p_3, r_{23}) = \nu_2 \cdot V'_{31}(r_{31}^2) \cdot (p_3 + p_1, r_{31}) = \\ = \nu_3 \cdot V'_{12}(r_{12}^2) \cdot (p_1 + p_2, r_{12}) \qquad (9)$$

where

$$\nu_i = \sqrt{m_i / \dot{x}_i^2}$$

By differentiating in τ and using the equations of motion, and iterating this procedure, we never find some other constraint, except for new relations involving the derivatives of the ν_i (that is new lagrangian constraints, but which do not imply new canonical constraints, since we cannot eliminate the ν_i in terms of the momenta p_i).

The relations (9) determine the ratios of the ν_i , and give no canonical secondary constraints at all.

From the point of view of Dirac's theory of constraints this corresponds to one first class constraint and two second class constraints. So this model cannot be predictive[13] . We would have five constraints in a lagrangian approach: 1 first class and 4 second class, or, equivalently, in the Todorov-Komar approach, 3 first class constraints and 2 gauge-fixing conditions τ -independent.

There is an exception to this situation: when one interaction between two particles is absent, that is when say the potential V_{13} is absent in eq. (5) . The corresponding configuration of the system would be that of an open chain, instead of a triangle. This case was studied in detail by Kamimura[18] .

But one property of the set (8) and (9) is interesting: it is the separability property, which is satisfied. We see by inspection that all possible clusters can be separated with the correct constraints, in a cyclic way. Each cluster will have the correct constraints for a two particle system or for a free particle, where the two particle system is described by the two body lagrangian (1) . This stems from the structure of the lagrangian (4), which is in itself separable.

This cluster decomposition property allows for possible configurations with two particles separated by a time-like distance, but be-

longing to the same cluster, like the chain configurations of the lagrangian model of reference[18]. This is not in contradiction with the definition of separability given in this lecture (see the appendix for the analysis of a particularly interesting case).

As an example of a cluster decomposition assume that the third particle goes to an infinite space-like distance from the others, and assume also that V_{31} and V_{32} go to zero with their derivatives. We get from eq. (8) and (9)

$$\left.\begin{aligned} \Omega_1 &= p_1^2 - m^2 + V_{12} \\ \Omega_2 &= p_2^2 - m^2 + V_{21} \\ \Omega_3 &= p_3^2 - m^2 \\ B &= V'_{12}(r_{12}^2)(p_1+p_2, r_{12}) \end{aligned}\right\} \qquad (10)$$

which is the set of constraints we expected for a two-body cluster (particles 1 and 2) and a free particle (particle 3).

So the lagrangian (4) doesn't give a correct set of constraints, but gives a suggestion of how to achieve the cluster decomposition property.

We may take advantage of that and postulate the following set of constraints for a separable model of three particles:

$$\Omega_i = p_i^2 - m^2 + \sum_{j\neq i}^{1,2,3} V_{ij}(r_{ij}^2) \quad ; \quad i = 1,2,3 \quad ; \quad V_{ij} = V_{ji} \qquad (11)$$

and

$$B_i = \sum_{j=1}^{3} V'_{ij}(r_{ij}^2)\cdot(p_i+p_j, r_{ij}) \qquad (12)$$

where in this last definition the subscript i may even take the value 3, since in this case eq. (12) gives a constraint already implied by B_1 and B_2.

From the point of view of the Todorov-Komar approach, the model (11), (12) can be reinterpreted in terms of three first class (among themselves) constraints and two scalar gauge-fixing conditions. These first class constraints do necessarily exist, at least locally. This should be clear from a general theorem that states that it is always possible (at least locally) to find a canonical transformation to a new set of variables, such that part of them are equivalent to the

constraints (in the sense that they locally define the same surface in the phase space); in this case three momenta, say, corresponding to the three first class constraints, and two coordinates, conjugated to two of these three momenta.

Contrary to the two-body case, in the three or N-body case it is in general a very difficult task to find explicitly the equivalent set of first class and gauge-fixing constraints. We know that it exists and so that a model based on eq. (11) and (12) has a counter-part in the Todorov-Komar approach.

The model (11) and (12) can be easily generalized to N-bodies if we use again a N-body lagrangian as a suggestion for the correct choice concerning separability.

Starting from

$$L = -\sum_{i=1}^{N} \sqrt{\mathcal{U}_i \dot{x}_i^2} \tag{13}$$

with

$$\mathcal{U}_i = m_i^2 - \sum_{j\neq i}^{1,\ldots,N} V_{ij}(r_{ij}^2) \quad ; \qquad V_{ij} = V_{ji} \tag{14}$$

we may infer the following set of 2N-1 constraints for the N-body problem:

$$\Omega_i = p_i^2 - m_i^2 + \sum_{j\neq i}^{1\ldots N} V_{ij}(r_{ij}^2) \quad ; \qquad i = 1, 2, \ldots, N \tag{15}$$

$$B_i = \sum_{j=1}^{N} V'_{ij}(r_{ij}^2)\cdot(p_i + p_j, r_{ij}) \tag{16}$$

where B_N is not independent, but it is implied by $B_1, \ldots, B_{N-1}$.

The set (15) and (16) has the cluster decomposition property, in the sense that it gives the corresponding set of constraints for each cluster when some coupling goes to zero. Even in the case of partial open chain configurations the correct set of constraints is reproduced.

As an example I will write eq. (16) for N = 4 :

$$\left.\begin{aligned} &V'_{12}(p_1+p_2, r_{12}) + V'_{13}(p_1+p_3, r_{13}) + V'_{14}\cdot(p_1+p_4, r_{14}) = 0 \\ &V'_{21}(p_1+p_2, r_{21}) + V'_{23}(p_2+p_3, r_{23}) + V'_{24}(p_2+p_4, r_{24}) = 0 \\ &V'_{31}(p_1+p_3, r_{34}) + V'_{32}(p_2+p_3, r_{32}) + V'_{34}(p_3+p_4, r_{34}) = 0 \end{aligned}\right\} \tag{17}$$

When for instance $V'_{i4} = 0$, we get three constraints, of which only two are independent, and these are the same as that of equation (12) for the particle 1,2 and 3.

One possible objection to this model is that the constraints Ω_i seem to contain only two-body forces, so they would not satisfy the separability requirement. But it must be observed that the Ω_i are not the mass constraints of the Todorov-Komar approach, as we have stressed before. It is the aim of the present note to give some preliminary result by showing a possible procedure to get from the Ω_i and B_i the first class constraints of the Todorov-Komar approach, at least in a particular case, that of a symmetric harmonic potential model. I will not exhibit completely these constraints, due to the algebraic complexity of the model, but it will result quite clearly that these new constraints are no more depending on two-body potentials only. I hope that the following analysis will give an insight into the complexity of the three-body problem, by indicating a possible explicit way to construct these first class constraints.

At the end I will give the generalization to the N-body problem.

As I have said I will consider the simplest case: that of three equal harmonic oscillators, so the set (11) and (12) becomes:

$$\left.\begin{aligned}
\Omega_1 &= p_1^2 - m^2 + \gamma (x_1-x_2)^2 + \gamma (x_1-x_3)^2 \\
\Omega_2 &= p_2^2 - m^2 + \gamma (x_2-x_1)^2 + \gamma (x_2-x_3)^2 \\
\Omega_3 &= p_3^2 - m^2 + \gamma (x_3-x_1)^2 + \gamma (x_3-x_2)^2 \\
B_1 &= (p_1+p_2,\, x_1-x_2) - (p_2+p_3,\, x_2-x_3) \\
B_2 &= (p_2+p_3,\, x_2-x_3) - (p_3+p_1,\, x_3-x_1)
\end{aligned}\right\} \qquad (18)$$

Let me perform the following linear canonical transformation

$$\left.\begin{aligned}
P &= p_1 + p_2 + p_3 \quad ; \quad X = \tfrac{1}{3}(x_1+x_2+x_3) \\
k_1 &= \tfrac{1}{6}(2p_1 - p_2 - p_3) \quad ; \quad r_1 = 2x_1 - x_2 - x_3 \\
k_2 &= \tfrac{1}{2}(p_2 - p_3) \quad ; \quad r_2 = x_2 - x_3
\end{aligned}\right\} \qquad (19)$$

with the inverse

$$\left.\begin{aligned} p_1 &= \tfrac{1}{3}p + 2k_1 \quad , \quad x_1 = x + \tfrac{1}{3}r_1 \\ p_2 &= \tfrac{1}{3}p - k_1 + k_2 \quad , \quad x_2 = x - \tfrac{1}{6}r_1 + \tfrac{1}{2}r_2 \\ p_3 &= \tfrac{1}{3}p - k_1 - k_2 \quad , \quad x_3 = x - \tfrac{1}{6}r_1 - \tfrac{1}{2}r_2 \end{aligned}\right\} \qquad (20)$$

By taking the following linear combinations of the constraints:

$$\left.\begin{aligned} \Omega_0 &= \Omega_1 + \Omega_2 + \Omega_3 \\ \phi_1 &= 2\Omega_1 - \Omega_2 - \Omega_3 \\ \phi_2 &= \Omega_2 - \Omega_3 \\ \psi_1 &= 3(B_1 + B_2) \\ \psi_2 &= B_2 - B_1 \end{aligned}\right\} \qquad (21)$$

I get

$$\left.\begin{aligned} \Omega_0 &= \tfrac{1}{3}p^2 - 3m^2 + \ell(3k_1^2 + k_2^2) + \gamma(r_1^2 + 3r_2^2) \\ \phi_1 &= 4(pk_1) + 8(3k_1^2 - k_2^2) + \tfrac{1}{2}\gamma(r_1^2 - 3r_2^2) \\ \phi_2 &= \tfrac{4}{3}(pk_2) - 4(k_1k_2) - \gamma(r_1r_2) \\ \psi_1 &= 2(pr_1) + 3(k_1r_1) - 3(k_2r_2) \\ \psi_2 &= 2(pr_2) - 3(k_1r_2) - (k_2r_1) \end{aligned}\right\} \qquad (22)$$

It can be verified, by calculating their Poisson algebra, that ϕ_i and ψ_i are IIth class constraints. We may take apart Ω_0, which contains the total mass, as an ingredient of the first class constraint, which could be evaluated following the prescription of Dirac.

In the symmetric model (all coupling constants equal to γ) the set (22) has an important property, which can be discovered by defining the new constraints:

$$\left.\begin{aligned} \chi_i &= \phi_i + \alpha\psi_i \\ \bar{\chi}_i &= \phi_i - \alpha\psi_i \end{aligned}\right\} \qquad (i = 1,2), \qquad (23)$$

where

$$\alpha = 2\sqrt{\gamma/3} \tag{24}$$

and the new canonical relative coordinates

$$\left.\begin{aligned} Q_1 &= \sqrt{\frac{2}{\alpha}}\left(k_1 + \frac{\alpha}{4} r_1\right) \quad , \quad P_1 = \sqrt{\frac{2}{\alpha}}\left(k_1 - \frac{\alpha}{4} r_1\right) \\ Q_2 &= \sqrt{\frac{2}{3\alpha}}\left(k_2 + \frac{3\alpha}{4} r_2\right) \quad , \quad P_2 = \sqrt{\frac{2}{3\alpha}}\left(k_2 + \frac{3\alpha}{4} r_2\right) \end{aligned}\right\} \tag{25}$$

We get

$$\left.\begin{aligned} \chi_1 &= \sqrt{2\alpha}\left[(p, 3Q_1 - P_1) + 3\sqrt{\frac{\alpha}{2}}\,(Q_1^2 - Q_2^2)\right] \\ \chi_2 &= \sqrt{\frac{2\alpha}{3}}\left[(p, 3Q_2 - P_2) - 6\sqrt{\frac{\alpha}{2}}\,(Q_1 \cdot Q_2)\right] \end{aligned}\right\} \tag{26}$$

and $\bar{\chi}_1, \bar{\chi}_2$ are obtained by χ_1 and χ_2 respectively, by the interchange of P_i with Q_i :

$$\left.\begin{aligned} \bar{\chi}_1 &= \sqrt{2\alpha}\left[(p, 3P_1 - Q_1) + 3\sqrt{\frac{\alpha}{2}}\,(P_1^2 - P_2^2)\right] \\ \bar{\chi}_2 &= \sqrt{\frac{2\alpha}{3}}\left[(p, 3P_2 - Q_2) - 6\sqrt{\frac{\alpha}{2}}\,(P_1 \cdot P_2)\right] \end{aligned}\right\} \tag{27}$$

Lastly Ω_0 becomes

$$\begin{aligned} \Omega_0 = \frac{1}{3} p^2 - 3m^2 + \frac{3\alpha}{4}\Big[3\left(Q_1^2 + Q_2^2 + P_1^2 + P_2^2\right) - \\ - 2\left((Q_1 P_1) + (Q_2 P_2)\right)\Big] \end{aligned} \tag{28}$$

We have the important property $\left(\{x_i^\mu, p_j^\nu\} = \delta_{ij} g^{\mu\nu}\right)$

$$\left.\begin{aligned} \{\chi_1, \chi_2\} &= 0 \\ \{\bar{\chi}_1, \bar{\chi}_2\} &= 0 \end{aligned}\right\} \tag{29}$$

This is the best simplification we may get by a linear canonical transformation, since in the Poisson algebra of the constraints only four independent quantities are present (with respect to six, which could be a priori present in a 4 x 4 antisymmetric matrix). But this simplifications is unfortunately possible in the symmetric case only.

The form (26), (27) and (28) of the constraints allows to see the

way to get three first class constraints (among themselves), which are in involution (with respect to the Poisson brackets) each other. It is only necessary to perform the following canonical transformations (which by the way are all in one of the Dirac classes, so they will give at the quantum level equivalent systems): firstly we transform to longitudinal and transverse parts with respect to p :

$$\begin{aligned} P_i^\mu &= -\epsilon_\lambda^\mu(p)\cdot\bar{P}_{i\lambda} + \frac{p^\mu}{\sqrt{p^2}}\cdot\bar{P}_i \\ Q_i^\mu &= -\epsilon_\lambda^\mu(p)\,\bar{Q}_{i\lambda} + \frac{p^\mu}{\sqrt{p^2}}\,\bar{Q}_i \end{aligned} \tag{30}$$

(where the ϵ_λ^μ are defined as in the Ith lecture), so we have

$$\begin{aligned} \{\bar{P}_i\,,\,\bar{Q}_j\} &= \delta_{ij} \\ \{\bar{P}_{i\lambda}\,,\,\bar{Q}_{j\lambda'}\} &= -\delta_{ij}\,\delta_{\lambda\lambda'} \end{aligned} \tag{31}$$

then we perform the phase transformation generated by the generating function

$$\begin{aligned} \psi(\bar{Q}_i\,,\bar{Q}_{i\lambda}) &= \frac{3}{2}(\bar{Q}_1^2+\bar{Q}_2^2) + \\ &+\sqrt{\frac{q\alpha}{2p^2}}\Big[\frac{1}{3}\bar{Q}_1^3 - \bar{Q}_2^2\bar{Q}_1 - \bar{Q}_1(\bar{Q}_{1\lambda}\bar{Q}_{1\lambda}-\bar{Q}_{2\lambda}\bar{Q}_{2\lambda}) + \\ &\qquad + 2\,\bar{Q}_2\,\bar{Q}_{1\lambda}\bar{Q}_{2\lambda}\Big] \end{aligned} \tag{32}$$

that is [+)]

$$\left.\begin{aligned} \pi_i &= e^{\psi}*\bar{P}_i = \bar{P}_i - \frac{\partial\psi}{\partial\bar{Q}_i} \\ \pi_{i\lambda} &= e^{\psi}*\bar{P}_{i\lambda} = \bar{P}_{i\lambda} + \frac{\partial\psi}{\partial\bar{Q}_{i\lambda}} \\ \rho_i &= e^{\psi}*\bar{Q}_i \equiv \bar{Q}_i \\ \rho_{i\lambda} &= e^{\psi}*\bar{Q}_{i\lambda} \equiv \bar{Q}_{i\lambda} \end{aligned}\right\} \tag{33}$$

so that we get

+) $e^A * B = B + \{A,B\} + \frac{1}{2}\{A,\{A,B\}\} + \dots$.

$$\left.\begin{aligned} \chi_1 &= -\pi_1 \\ \chi_2 &= -\pi_2 \end{aligned}\right\} \tag{34}$$

In this way we put two of the constraints in form to two canonical coordinates. At this point we should evaluate the two conjugated variables ρ_1 and ρ_2 from the equations

$$\bar{\chi}_1 = 0 \quad , \quad \bar{\chi}_2 = 0 \tag{35}$$

and substitute the two constraints $\bar{\chi}_i$ with the equivalent ones

$$\left.\begin{aligned} \tilde{\chi}_1 &= \rho_1 - f_1(\pi_i, \pi_{i\lambda}, \rho_{i\lambda}) \\ \tilde{\chi}_2 &= \rho_2 - f_2(\pi_i, \pi_{i\lambda}, \rho_{i\lambda}) \end{aligned}\right\} \tag{36}$$

which have again zero Poisson bracket $\{\tilde{\chi}_1, \tilde{\chi}_2\} = 0$, and are variables conjugated to π_1 and π_2.

By putting $\pi_i = 0$, $\rho_i = f_i(0, \pi_{i\lambda}, \rho_{i\lambda})$ in Ω_0 we will get a new constraint $\tilde{\Omega}_0 = \tilde{\Omega}_0(\pi_{i\lambda}, \rho_{i\lambda})$ which certainly commutes with χ_1 and χ_2, so giving a set of three first class constraints (among themselves).

Unfortunately we cannot solve analytically the two equations $\bar{\chi}_i = 0$, which are two algebraic equations in ρ_i of the 4th degree, so the program to get the complete canonical transformation cannot be performed explicitly; nevertheless what we should do is now quite clear, and the difficulties are quite circumscribed.

To complete the program we should perform a last canonical transformation (suggested to me by K. Kamimura):

$$\left.\begin{aligned} \tilde{\pi}_i &= e^{\pi_\ell f_\ell} * \pi_i \equiv \pi_i \\ \tilde{\rho}_i &= e^{\pi_\ell f_\ell} * \rho_i = \rho_i - f_i(\pi_i, \pi_{i\lambda}, \rho_{i\lambda}) \\ &\text{etc.} \end{aligned}\right\} \tag{37}$$

which will bring all the four second class constraints χ_i and $\bar{\chi}_i$ into canonical variables. Fortunately, as suggested to me by K. Kamimura, it is always possible to eliminate between the three equations (of the 4th degree in ρ_i)

$$\overline{\chi}_1 = 0 \quad , \quad \overline{\chi}_2 = 0 \quad , \quad \Omega_0 = 0 \tag{38}$$

the two variables ρ_1 , ρ_2 by using, for instance, the Bezout's elimination method, is obtaining a constraint which will not depend on ρ_i and where we may freely put $\pi_i = 0$, which will be third first class constraint beside χ_1 and χ_2 (it will be an algebraic equation of very high degree in p^2).

I did not do this elimination explicitly, which corresponds to the calculation of a very big determinant (a 45×45 determinant for $N = 3$), but it is important in my opinion to realize that it is always possible by using linear methods only, and even in the N-body case.

For the N-body case the generalization of equations (19) and (20) is the following

$$\left.\begin{aligned} r_i &= (N-i)\,x_i - (x_{i+1} + \ldots + x_N) \quad ; \quad i = 1, 2, \ldots, N-1 \\ r_N &= x = \frac{1}{N}(x_1 + \ldots + x_N) \\ k_i &= \left[(N-i)\,p_i - (p_{i+1} + \ldots + p_N)\right] \Big/ (N-i)(N-i+1) \\ k_N &= p = p_1 + p_2 + \ldots + p_N \end{aligned}\right\} \tag{39}$$

and that of equations (23), (24) and (25) is

$$\left.\begin{aligned} \chi_i &= \phi_i + \alpha\,\psi_i \\ \overline{\chi}_i &= \phi_i - \alpha\,\psi_i \end{aligned}\right. \quad , \quad i = 1, 2, \ldots, N-1 \tag{40}$$

where

$$\left.\begin{aligned} Q_i &= \sqrt{\frac{(N-i)(N-i+1)}{N\alpha}}\left[\, k_i + \frac{N\alpha}{2(N-i)(N-i+1)}\, r_i \,\right] \\ P_i &= \sqrt{\frac{(N-i)(N-i+1)}{N\alpha}}\left[\, k_i - \frac{N\alpha}{2(N-i)(N-i+1)}\, r_i \,\right] \end{aligned}\right\} \tag{41}$$

and

$$\alpha = 2\sqrt{\frac{T}{N}} \tag{42}$$

Finally the set of constraints χ_i , $\bar{\chi}_i$ so defined is the following

$$\chi_i = \sqrt{\frac{(N-i)(N-i+1)}{N}}\,(P, 3Q_i - P_i) +$$

$$+ \alpha N (N-i-1) Q_i^2 - \alpha N (Q_{i+1}^2 + \dots + Q_{N-1}^2) +$$

$$- 2\alpha N \sqrt{(N-i)(N-i+1)}\left(Q_i, \frac{Q_1}{\sqrt{(N-1)N}} + \frac{Q_2}{\sqrt{(N-2)(N-1)}} + \dots\right.$$

$$\left. + \dots + \frac{Q_{i-1}}{\sqrt{(N-i+1)(N-i+2)}}\right) \; ; \qquad i = 1, 2, \dots, N-1 \qquad (43)$$

$$\bar{\chi}_i = \chi_i \qquad \text{with} \qquad Q_i \longleftrightarrow P_i \qquad (43^*)$$

They enjoy the property

$$\{\chi_i, \chi_j\} = 0 \quad ; \quad \{\bar{\chi}_i, \bar{\chi}_j\} = 0 \qquad (44)$$

APPENDIX

In this appendix I want to discuss in a simple case the existence of solutions of the constraint equations (11) and (12) . In particular the constraints (12) will give some restriction on the character of the distances between the particles, $r_{ij} = x_i - x_j$, which will depend on the potentials through V'_{ij} .

Let me consider a choice of the potentials such that $V_{ij}(r^2_{ij}) = V(r^2_{ij})$, with $V'(r^2) > 0$ for any value of the argument, and such that $V(r^2) \to 0$, $V'(r^2) \to 0$ for $r^2 \to -\infty$. This means that for large space-like distances between two particles they become not interacting, while for a time-like distance they interact.

The limitation $V'_{ij} = V' > 0$ makes more easy the discussion of the implications of the constraints (12).

Another requirement on V is that p_i^2 be always positive. This can be obtained by requiring that V be bound from above by a value V_o such that $2V_o < m^2$.

With this choice we may find a limitation on the values of r^2_{ij} due to equations (12). By some algebraic calculation it is possible to show that we have:

$$\left.\begin{aligned} r_{12}^2 + r_{23}^2 &< 0 \\ r_{23}^2 + r_{31}^2 &< 0 \\ r_{31}^2 + r_{12}^2 &< 0 \end{aligned}\right\} \qquad \text{(A1)}$$

The boundary of the region defined by eq. (A1) is not such that r^2_{ij} is always < 0 , for any value of i and j . Configurations with <u>one</u> distance time-like and <u>two</u> space-like are allowed (but not two time-like).

The existence of such configurations does not contradict separability as defined at the beginning of this lecture, indeed, when the two space-like distances become infinite, and one particle becomes free, the remaining cluster of two particles has the correct set of constraints, so the distance which originally was time-like is forced by the constraint to become space-like.

Nothing can be said about the possible existence of bound states when the system is in such exceptional configurations, without solving explicitly the motion for a given potential. But this is a difficult task which will deserve further investigation.

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(+) I will assume $\dot{x}_i^2 > 0$ and $U_i > 0$ in order to avoid the existence of faster than lighth particles.

THE HAMILTON-JACOBI FORMALISM FOR SYSTEMS WITH CONSTRAINTS

G. Longhi
Istituto di Fisica Teorica
Universita di Firenze
Sezione Istituto Nazionale di Fisica Nucleare di Firenze

In collaboration with

D. Dominici
Sezione I.N.F.N di Firenze

and

J. Gomis, J.M. Pons
Departamento de Física Teórica
Universidad de Barcelona

1. Introduction

In this note we will study a system described in a given phase space x^i , p_i , $i = 1,2,\ldots,n$, by a set of constraints both of first and second class:

$$\Omega_\rho (x^i, p_i) = 0 \qquad \rho = 1, \ldots, p \leq n \qquad (1.1)$$

$$\chi_\kappa (x^i, p_i) = 0 \qquad \kappa = 1, \ldots, 2s \leq 2n \qquad (1.2)$$

where Ω_ρ are first class and χ_κ of second class (so the χ_κ are even in number).

Moreover we assume that the canonical hamiltonian is identically zero, so the dynamics is all contained in the equations (1.1) and (1.2).

Let us stress that any system can be studied within this scheme, since it is always possible to substitute any given lagrangian, not singular, with a new lagrangian which depends on one more coordinate (and velocity), homogeneous of first degree in the new velocities, so giving rise to a canonical hamiltonian identically zero (see for instance H. Rund; see bibliographical note).

By hypothesis we will have:

$$\{\Omega_\rho , \Omega_\sigma\} \equiv \sum_\tau C_{\rho\sigma\tau}\,\Omega_\tau \tag{1.3}$$

$$\{\Omega_\rho , \chi_k\} \equiv \sum_\sigma C_{\rho k\sigma}\,\Omega_\sigma + \sum_h C_{\rho kh}\,\chi_h \tag{1.4}$$

2. First Class Constraints

Let us study at first the case when the χ_k are absent. So our system will be described by the set (1.1).

We want to show that in this case (1.1) can be considered as a set of partial differential equations of first order, in general non linear, in one unknown function, $S = S(x^i)$, the Hamilton-Jacobi function:

$$\Omega_\rho \left(x^i, \frac{\partial S}{\partial x^i}\right) = 0 \tag{2.1}$$

with

$$p_i = \frac{\partial S}{\partial x^i} = f_i(x) \tag{2.2}$$

where the $p_i = f_i(x)$ identically satisfy equations (1.1).

We will demonstrate this in three steps:

i) One shows that the system of equations (2.1) is complete, as a consequence of the first class character of the Ω_ρ , and that it can always, at least locally, transformed into a set of equations in involution (that is in a set of equations such that their l.h.s. have zero Poisson brackets each other, identically, and not only on the surface in phase space defined by the constraints).

ii) Following the Jacobi's method of integration one can extend the set (when $p < n$, otherwise if $p = n$ this step is unnecessary) to a set of n equations in involution, which will contain $n - p$ arbitrary constants. It is shown that this is always possible. This set of equations is solved in the momenta: $p_i = f_i(x^i, a_a)$. $i = 1,\ldots,n$; $a = p+1,\ldots, n$, where a_a are the arbitrary constants. This solution allows us to find the form $dS = f_i(x,a)\,dx^i$.

iii) One shows that

$$\frac{\partial f_i}{\partial x^j} = \frac{\partial f_j}{\partial x^i}$$

so the 1-form dS is exact, and a function $S = S(x,a)$ exists.

i) Let us assume that the equations $\Omega_\rho(x^i,p_i) = 0$ are independent in the p_i, this means that we may assume

$$\operatorname{rank}\left\| \frac{\partial \Omega_\rho}{\partial p_i} \right\| = p \tag{2.4}$$

(If we assume that Ω_ρ are a set of independent functions, that is that no relation between the Ω_ρ exists, it follows that, apart from a possible canonical transformation which exchanges some of the x^i in an equal number of p_i, the rank of the matrix $\|\partial\Omega_\rho/\partial p_i\|$ is equal to p).

From (2.4) it follows that we can solve the equations (1.1) in terms of, say p_ρ, $\rho = 1,\dots,p$:

$$p_\rho = \psi_\rho(x^i,p_a) \quad ; \qquad a = p+1,\dots,n \tag{2.5}$$

The set of equations

$$F_\rho(x,p) = p_\rho - \psi_\rho(x^i,p_a) = 0 \tag{2.6}$$

is locally equivalent to the set (1.1), in the sense that it defines locally the same surface in the phase space.

Now, as a consequence of the first class character of the Ω_ρ, the F_ρ are in involution. Indeed

$$\{p_\rho - \psi_\rho(x^i,p_a),\ p_\sigma - \psi_\sigma(x^i,p_a)\}$$

does not depend on p_ρ, $\rho = 1,\dots,p$:

$$\{F_\rho, F_\sigma\} = \frac{\partial \psi_\sigma}{\partial x^\rho} - \frac{\partial \psi_\rho}{\partial x^\sigma} + \{\psi_\rho, \psi_\sigma\} \quad ; \qquad \{x^i,p_j\} = \delta^i_j \tag{2.7}$$

so in order to show that it is zero it is sufficient to show that it is so when $p_\rho = \psi_\rho(x,p_a)$.

Now from (2.5) it follows identically in x^i and p_a :

$$\Omega_\rho(x^i, \psi_\rho(x^i,p_a), p_a) = 0$$

from which we get

$$\frac{\partial \Omega_\rho}{\partial x^i} + \frac{\partial \Omega_\rho}{\partial p_\sigma} \cdot \frac{\partial \psi_\sigma}{\partial x^i} \approx 0 \tag{2.8}$$

$$\frac{\partial \Omega_\rho}{\partial p_a} + \frac{\partial \Omega_\rho}{\partial p_\sigma} \cdot \frac{\partial \psi_\sigma}{\partial p_a} \approx 0 \qquad (2.9)$$

where $\approx$ means that they are equal to zero when in the partial derivatives of Ω_ρ we substitute ψ_ρ for p_ρ .

The (2.8) and (2.9) can be rewritten:

$$\frac{\partial \Omega_\rho}{\partial x^i} \approx \frac{\partial \Omega_\rho}{\partial p_\sigma} \cdot \frac{\partial (p_\sigma - \psi_\sigma)}{\partial x^i} \qquad (2.10)$$

$$\frac{\partial \Omega_\rho}{\partial p_i} \approx \frac{\partial \Omega_\rho}{\partial p_\sigma} \cdot \frac{\partial (p_\sigma - \psi_\sigma)}{\partial p_i} \qquad (2.11)$$

since the last equations for $i = 1,\ldots,p$ become identities .

From (2.10) and (2.11) it follows

$$\{\Omega_\rho , \Omega_\sigma\} = \frac{\partial \Omega_\rho}{\partial p_\tau} \cdot \frac{\partial \Omega_\sigma}{\partial p_\varepsilon} \{p_\tau - \psi_\tau , p_\varepsilon - \psi_\varepsilon\} \qquad (2.12)$$

where $\tau, \varepsilon = 1,\ldots, p$

Now the l.h.s. is weakly zero, so that if we put

$$Z_\tau^{(\sigma)} = \frac{\partial \Omega_\sigma}{\partial p_\varepsilon} \{p_\tau - \psi_\tau , p_\varepsilon - \psi_\varepsilon\}$$

we will have

$$\frac{\partial \Omega_\rho}{\partial p_\tau} \cdot Z_\tau^{(\sigma)} \approx 0$$

which is a linear homogeneous set of equations in $Z_\tau^{(\sigma)}$, with the determinant of the coefficients different from zero by hypothesis. So it follows

$$Z_\tau^{(\sigma)} \approx 0$$

By iterating the argument we also get

$$\{p_\rho - \psi_\rho , p_\sigma - \psi_\sigma\} \approx 0$$

from which it follows that

$$\{F_\rho , F_\sigma\} = \{p_\rho - \psi_\rho , p_\sigma - \psi_\sigma\} = 0 \qquad (2.13)$$

on the entire phase space an not only when $p_\rho = \psi_\rho(x^i, p_a)$.

The set of equations (1.1) is complete in the sense used in the theory of system of partial differential equations. We have shown that it is locally equivalent to a set in involution. So the step (i) is

demonstrated.

ii) The set F_ρ can be always extended to a set of n functions in involution (if $p < n$, otherwise this step is unnecessary):

$$F_i(x^i, p_i) \quad , \qquad i = 1, \ldots, n \tag{2.14}$$

indeed, if we consider the equations in G

$$\{F_\rho, G\} = 0 \qquad , \rho = 1, \ldots, p \tag{2.15}$$

they are a set of homogeneous partial differential equations of the first order in G, which are in involution. Indeed if we put

$$X_\rho(G) = \{F_\rho, G\} \tag{2.16}$$

we have

$$X_\rho(X_\sigma(G)) - X_\sigma(X_\rho(G)) =$$

$$= \{F_\rho, \{F_\sigma, G\}\} - \{F_\sigma, \{F_\rho, G\}\} = \{\{F_\rho, F_\sigma\}, G\} \tag{2.17}$$

as a consequence of the Jacobi's identity and of equation (2.13).

From a theorem on homogeneous linear systems of partial differential equations, we know that the system (2.15) has in this case (complete system in jacobian form) n-p independent solutions in involution among themselves besides the F_ρ , G_a , $a = p+1, \ldots, n$.

If we now choose a G_a we have a new set of equations as (2.15), with G_a added to the F_ρ , that is a system of p+1 equations of the same kind. We can therefore iterate the procedure until we will have n functions in involution; at this point the procedure will stop, since a set of n equations of this kind has a constant as only solution.

It is so demonstrated the possibility of extending (in infinite ways) the set F_ρ to a set of n functions in involution, $F_i(x^i, p_i)$, $i = 1, \ldots, n$.

Since adding to the F_i some constant their involutory character is not altered, we may put

$$\left.\begin{aligned} F_\rho(x^i, p^i) &= 0 \quad , \quad \rho = 1, \dots p \\ F_a(x^i, p_i) &= c_a \quad , \quad a = p+1, \dots, n \end{aligned}\right\} \tag{2.18}$$

where c_a are n-p arbitrary constants; we will write

$$F_i(x^i, p_i) = c_i \quad , \quad i = 1, \dots, n \tag{2.19}$$

with

$$c_\rho = 0 \qquad \rho = 1, \dots p \tag{2.20}$$

Since the functions F_i so determined are independent by construction, apart from a possible reordering of the canonical variables (by eventually performing a canonical transformation which interchanges some x^i with some p_i) , we may assume

$$\left| \frac{\partial F_i}{\partial p_j} \right| \neq 0 \tag{2.21}$$

It then follows that we may solve the equations

$$F_i(x, p) = c_i$$

in the form

$$p_i - f_i(x^i, c_i) = 0 \quad , \qquad i = 1, \dots n \tag{2.22}$$

By using the same arguments as in (i), we find that the new set

$$P_i = p_i - f_i(x, c) \tag{2.23}$$

is in involution:

$$\{P_i, P_j\} = 0 \tag{2.24}$$

iii) From the equation (2.24) it follows immediately

$$\frac{\partial f_i}{\partial x^j} - \frac{\partial f_j}{\partial x^i} = 0 \tag{2.25}$$

so the 1-form

$$dS = f_i(x, c) \cdot dx^i \tag{2.26}$$

is exact.

It follows that, in the hypothesis that the set of constraints Ω_ρ is of first class, it exists a function

$$S = S(x,c) \tag{2.27}$$

such that the

$$p_i = \frac{\partial S}{\partial x^i} \tag{2.28}$$

are solutions of the equations (Hamilton-Jacobi's equations)

$$\Omega_\rho \left(x^i, \frac{\partial S}{\partial x^i}\right) = 0 \tag{2.29}$$

The function S is defined by (2.26) apart for an unessential additive constant. Neglecting this constant S will in general contain n-p arbitrary constants of integration.

The constraints are contained implicitely in the set of equations

$$P_i = p_i - f_i(x,c) = 0 \tag{2.30}$$

indeed by eliminating the n-p constants c_a from this set of equations, we get again the original constraints (1.1).

The solution S so obtained contains as many constans as the number of variables x^i minus the number of constraints Ω_ρ. So it is a <u>complete integral</u>, and it is known from the theory of systems of partial differential equations of the first order, that from a complete integral it is possible to get all other integrals (the singular and the general integral) by means of differentiations and eliminations only. So the Jacobi's method of integration gives a general kind of solution.

When second class constraints are present, the set of equations $\Omega_\rho = 0$, $\chi_k = 0$ cannot be interpreted as a set of Hamilton-Jacobi equations.

We will see in Section 4 what can be said in this case.

Before concluding this section, let us observe that the transformation

$$\left.\begin{aligned} Q_i &= x_i \\ P_i &= p_i - f_i(x,c) \end{aligned}\right\} \tag{2.31}$$

is a canonical transformation, due to equations (2.24) or (2.25). It is a phase transformation, generated by the function $-S(x,c)$:

$$\left.\begin{aligned} Q_i &= e^{-S} * x_i = x_i \\ P_i &= e^{-S} * p_i = p_i - f_i \end{aligned}\right\} \tag{2.32}$$

where the operation $*$ is defined by

$$e^A * B = B + \{A,B\} + \tfrac{1}{2}\{A,\{A,B\}\} + \dots$$

3. Equations of motion

The first class constraints Ω_ρ are not in involution in general (the r.h.s. of eq. (1.3) is not identically zero in general), so that the adjoint system (characteristic system) associated to them, written in parametric form, as in the following equation (3.2), is in general not integrable. We must substitute to the Ω_ρ the F_ρ :

$$F_\rho(x,p) = p_\rho - \psi_\rho(x^i, p_a) = 0 \tag{3.1}$$

The adjoint system associated to the set (3.1) is given by

$$\left.\begin{aligned} dx^i &= \{x^i, F_\rho\}\, d\theta^\rho \quad ; \quad i = 1,\dots n \\ dp_i &= \{p_i, F_\rho\}\, d\theta^\rho \quad ; \quad \rho = 1,\dots p \end{aligned}\right\} \tag{3.2}$$

where the parameters θ^ρ are defined by these same equations.

From the first set of equations we have for $i = \sigma = 1,\dots,p$:

$$dx^\sigma = \{x^\sigma,\ p_\rho - \psi_\rho(x^i, p_a)\}\, d\theta^\rho = d\theta^\sigma$$

so we can choose

$$x^\rho = \theta^\rho \tag{3.3}$$

From the other equations we have

$$dp_\rho = -\{p_\rho, \psi_\sigma\}\, dx^\sigma$$

$$dx^a = -\{x^a, \psi_\sigma\}\, dx^\sigma$$

$$d p_a = - \{ p_a , \psi_\sigma \} dx^\sigma \tag{3.4}$$

so the "hamiltonians" which generate the evolution in the "times" x^σ are $-\psi_\sigma$.

From (3.2) it follows

$$d f(x,p,\theta) = \left[\frac{\partial f}{\partial \theta^\rho} + \{ f , F_\rho \} \right] d\theta^\rho \tag{3.5}$$

It is easily verified that the system (3.2) is integrable

$$\frac{\partial^2 x^i}{\partial \theta^\lambda \partial \theta^\rho} = \frac{\partial}{\partial \theta^\lambda} \{ x^i , F_\rho \} = \{ \{ x^i , F_\rho \} , F_\lambda \} =$$

$$= \{ \{ x^i , F_\lambda \} , F_\rho \} + \{ x^i , \{ F_\rho , F_\lambda \} \} =$$

$$= \frac{\partial^2 x^i}{\partial \theta^\rho \partial \theta^\lambda}$$

and so on.

It even easily verified that, if the x^i , p_i satisfy the equations (3.2), they also satisfy the equations $F_\rho = 0$, and so the equations $\Omega_\rho = 0$. Indeed

$$d F_\rho (x,p) = \{ F_\rho , F_\lambda \} \, d\theta^\lambda = 0 \tag{3.6}$$

so that, when for some value of the θ^ρ it is $F_\rho = 0$, they remain so for any other value of the parameters, that is for any value of x^i , p_i belonging to the same characteristic strip determined by eq. (3.2).

Equation (3.6) can be called the <u>stability condition</u> for the F_ρ . The same holds for the Ω_ρ . Indeed when $\Omega_\rho = 0$, it is

$$\left. \begin{aligned} \partial \Omega_\rho / \partial x^i &= \left(\partial \Omega_\rho / \partial p_\sigma \right) \cdot \left(\partial F_\sigma / \partial x^i \right) \\ \partial \Omega_\rho / \partial p_i &= \left(\partial \Omega_\rho / \partial p_\sigma \right) \cdot \left(\partial F_\sigma / \partial p_i \right) \end{aligned} \right\} \tag{3.7}$$

see eq. (2.10) and (2.11). It follows

$$d \Omega_\rho = \frac{\partial \Omega_\rho}{\partial p_\sigma} \cdot d F_\sigma \tag{3.8}$$

from which it follows the stability of the constraints Ω_ρ .

We want now show that the solutions of the equations of motion

(3.2) (hamilton equations of motion) are the same (apart from an identification of the constants of integration) as that obtained from the Jacobi's method; that is, given a complete integral of the H-J equations

$$S = \phi(x,c) + const. \tag{3.9}$$

the solutions are

$$p_i = \frac{\partial \phi}{\partial x^i} \quad , \qquad i = 1, \dots, n \tag{3.10}$$

$$b^a = \frac{\partial \phi}{\partial c_a} \quad , \qquad a = p+1, \dots, n \tag{3.11}$$

where b^a are new arbitrary constants, together with the equations of the constraints (3.1).

It is always possible to introduce p parameters τ^ρ in order to express the equations (3.10) and (3.11) in the following parametric form

$$\left.\begin{aligned} x^i &= x^i(c, b, \tau) \\ p_i &= p_i(c, b, \tau) \end{aligned}\right\} \tag{3.12}$$

I will assume that the parameters τ^ρ are essential, that is that

$$\operatorname{rank} \left\| \frac{\partial x^i}{\partial \tau^\rho} \right\| = p \tag{3.13}$$

If we differentiate the eq. (3.11) with respect to τ^ρ, keeping b^a and c_a constant, we have

$$\frac{\partial}{\partial \tau^\rho}\left(\frac{\partial \phi}{\partial c_a}\right) = \frac{\partial^2 \phi}{\partial x^i \partial c_a} \cdot \frac{\partial x^i}{\partial \tau^\rho} = 0 \tag{3.14}$$

since $\phi(x,c)$ depends on τ^ρ only through the x^i. On the other hand, if we differentiate eq. (3.1), where $p_i = \partial\phi/\partial x^i$, and where the dependence on c_a is only through the function ϕ, with respect to c_a, we get

$$\frac{\partial F_\rho}{\partial p_i} \cdot \frac{\partial^2 \phi(x,c)}{\partial c_a \partial x^i} = 0 \tag{3.15}$$

Now the rank of the matrix $\left\| \partial^2\phi / \partial c \cdot \partial x \right\|$ is n-p (this fact stems from the completeness of the integral ϕ; it is indeed the precise mathematical definition of a complete integral), so between the solutions

of the two systems (3.14) and (3.15) will hold a linear relation:

$$\frac{\partial F_\rho}{\partial p_i} = C_\rho^\sigma \cdot \frac{\partial x^i}{\partial \tau^\sigma} \tag{3.16}$$

where in general C_ρ^σ will depend on c_a, b^a and τ^ρ (see the Appendix for a demostration of (3.16). We can at this point define a new set of parameters θ^ρ such that

$$C_\rho^\sigma \frac{\partial \theta^{\rho'}}{\partial \tau^\sigma} = \delta_\rho^{\rho'} \tag{3.17}$$

and put (3.16) in the form

$$\frac{\partial x^i}{\partial \theta^\rho} = \frac{\partial F_\rho}{\partial p_i} \tag{3.18}$$

which is the first set of the equations (3.2).

To get the second set of equations (3.2) we may differentiate in τ^ρ the equations (3.10)

$$\frac{\partial p_i}{\partial \tau^\rho} = \frac{\partial^2 \phi}{\partial x^{i'} \partial x^i} \cdot \frac{\partial x^{i'}}{\partial \tau^\rho} \tag{3.19}$$

and differentiating in x^i the equations (3.1) we get

$$\frac{\partial F_\rho}{\partial x^i} + \frac{\partial F_\rho}{\partial p_j} \cdot \frac{\partial^2 \phi}{\partial x^{i'} \partial x^i} \tag{3.20}$$

or, using (3.16) :

$$\frac{\partial F_\rho}{\partial x^i} + C_\rho^\sigma \cdot \frac{\partial x^j}{\partial \tau^\sigma} \cdot \frac{\partial^2 \phi}{\partial x^{i'} \partial x^i} = \frac{\partial F_\rho}{\partial x^i} + C_\rho^\sigma \cdot \frac{\partial p_i}{\partial \tau^\sigma} = 0 \tag{3.21}$$

or, with (3.17)

$$\frac{\partial p_i}{\partial \theta^\rho} = - \frac{\partial F_\rho}{\partial x^i} \tag{3.22}$$

which are the second set of eq. (3.2) .

Now the equations (3.2) have solutions which will depend on $2(n-p)$ constants of integration, since by eliminating the parameters τ^ρ we will get $n-p$ total differential equations for dx^i and $n-p$ for dp_i ; $2(n-p)$ is the number of constants that appear in the Jacobi's solution (3.10) and (3.11).

This shows that the solutions (3.10) and (3.11) are the same as that obtained from the hamilton equations of motion (3.2).

4. Second Class Constraints

Let us consider the set of constraints of both first and second class (1.1) and (1.2)

$$\left.\begin{aligned} \Omega_\rho(x,p) &= 0 \quad , \qquad \rho = 1,\dots, p < n \\ \chi_\ell(x,p) &= 0 \quad , \qquad \ell = 1,\dots, 2s \end{aligned}\right\} \tag{4.1}$$

where the second class constraints χ_ℓ are even in number.

Let us assume that

$$p+s \leq n \tag{4.2}$$

otherwise the procedure we will describe in the present Section is not valid.

It is in general possible to substitute the set (4.1) (at least locally) with a new set such that p+s constraints are of first class among themselves, and the remaining s are all of second class (see for instance S. Shanmugadhasan, Journ. Math. Phys. 14,677(1973), where the condition (4.2) is implied, and references quoted therein). So let me suppose that Ω_ρ and χ_ℓ with $\ell = 1,\dots,s$ are this subset of first class constraints among themselves. Let me call the remaining χ_ℓ with $\ell = s+1,\dots,2s$, Θ_k :

$$\Theta_k = \chi_{s+k} \quad , \qquad k = 1,\dots,s \tag{4.3}$$

For what we have said in Section 2, it will exists a function S = S(x,c) for the set of first class constraints Ω_ρ, χ_ℓ ($\ell = 1,\dots,s$) :

$$S = \phi(x^i, c_a) + \text{const.} \tag{4.4}$$

with $i = 1,\dots,n$, and $a = p+x+1,\dots,n$. The $n - (p+s)$ constants c_a are arbitrary constants of integration. The additive constant is not essential and may be neglected. The S is such that

$$p_i = \frac{\partial S}{\partial x^i} = \frac{\partial \phi(x,c)}{\partial x^i} \tag{4.5}$$

and these functions of x^i satisfy the equations

$$\Omega_\rho(x,p) = 0 \quad , \quad \rho = 1,\dots,p$$

$$\chi_\ell(x,p) = 0 \quad , \qquad \ell = 1,\dots,s \tag{4.6}$$

identically with respect to c_a .

Let me substitute the constraints Ω_ρ and χ_ℓ of (4.6) with the equivalent set of constraints in involution

$$F_\mu(x,p) = 0 \quad , \qquad \mu = 1,\dots,p+s \tag{4.7}$$

The Hamilton's equations of motion (characteristic system) associated to the F_μ will be:

$$\left.\begin{aligned} dx^i &= \{x^i, F_\mu\}\, d\theta^\mu \\ dp_i &= \{p_i, F_\mu\}\, d\theta^\mu \end{aligned}\right\} \tag{4.8}$$

where the p+s parameters θ^μ are defined by the (4.8) themselves. Due to the <u>involutive character</u> of the F_μ the system (4.8) is <u>integrable</u>.

It useful at this point to introduce the tensorial notation

$$\{\xi^\alpha\} = (x^1,\dots,x^n;\ p_1,\dots,p_n) \tag{4.9}$$

$$\alpha = 1, 2, \dots n$$

and the tensor

$$\| \epsilon^{\alpha\beta} \| = \begin{bmatrix} 0 & E \\ -E & 0 \end{bmatrix} \tag{4.10}$$

where E is the unit matrix n x n .

The equations (4.8) can now be written

$$d\xi^\alpha = \{\xi^\alpha, F_\mu\}\, d\theta^\mu \tag{4.11}$$

where

$$\{\xi^\alpha, F_\mu\} = \epsilon^{\alpha\beta} \frac{\partial F_\mu}{\partial \xi^\beta} \tag{4.12}$$

If we now require that the solutions of the equations (4.11) must satisfy the further conditions (constraints (4.3)) :

$$\Theta_\kappa(\xi) = 0 \tag{4.13}$$

we may think at this condition as a restriction on the parameters θ^μ. This means that it must hold

$$\Theta_\kappa\left(\xi^\alpha(\theta^\mu)\right) = 0 \tag{4.14}$$

or, for the $d\theta^\mu$:

$$\frac{\partial \Theta_\kappa(\xi)}{\partial \xi^\alpha} \cdot \frac{\partial \xi^\alpha}{\partial \theta^\mu} \cdot d\theta^\mu = 0 \tag{4.15}$$

Let us put

$$B^\kappa_{\cdot\mu} = \frac{\partial \Theta_\kappa(\xi)}{\partial \xi^\alpha} \cdot \frac{\partial \xi^\alpha}{\partial \theta^\mu} \tag{4.16}$$

which can also be written (see eq. (4.11))

$$B^\kappa_{\cdot\mu} = \frac{\partial \Theta_\kappa(\xi)}{\partial \xi^\alpha} \cdot \{\xi^\alpha, F_\mu\} = \{\Theta_\kappa, F_\mu\} \tag{4.17}$$

The $d\theta^\mu$ will be restricted by

$$B^\kappa_{\cdot\mu}\, d\theta^\mu = 0 \tag{4.18}$$

This is a set of total differential equations for the θ^μ, which can be thought as the adjoint system associated to the linear homogeneous system of equations in the unknown function $u = u(\theta)$:

$$\bar{B}^\mu_{\cdot\rho} \cdot \frac{\partial u}{\partial \theta^\mu} = 0 \tag{4.19}$$

where

$$B^\kappa_{\cdot\mu} \cdot \bar{B}^\mu_{\cdot\rho} = 0 \tag{4.20}$$

A solution of the set of equations (4.18) is thus

$$d\theta^\mu = \bar{B}^\mu_{\cdot\rho}\, d\omega^\rho \qquad , \quad \rho = 1, \ldots, p \tag{4.21}$$

where the ω^ρ are new p independent parameters.

In other words, the conditions (4.13) on the solutions of the Hamilton's equations of motion reduce the number of independent parameters from p+s to p .

Let me observe that the system (4.11) plus (4.13) is not an integrable system of the mixed kind (see for instance L.P. Eisenhart, "Continuous Groups of Transformations", pag. 4, or T. Levi-Civita,

"The absolute Differential Calculus", pag. 29 and foll.) . Indeed we know that the original set of constraints is not an integrable system of equations in some function S , such that $\partial S/\partial x^i = p_i$. Nevertheless we can consider the conditions $\Theta_\kappa = 0$ as supplementary conditions on the solutions of the integrable set of equations $F_\mu = 0$.

It remains to calculate the coefficients $\overline{B}^\mu_{\cdot\rho}$. The coefficients $B^\kappa_{\cdot\mu}$ defined in (4.17) constitute a rectangular matrix $s \times (p+s)$. This matrix has certainly a rank = s , since we know that the second class constraints χ_ℓ , $\ell = 1,\dots,2s$, have the property that

$$|\{\chi_\ell, \chi_{\ell'}\}| \neq 0$$

by relabeling, if necessary, the constraints F_μ , I may assume that the second class constraints of the set F_μ, Θ_κ are

$$F_{p+\kappa} \ , \Theta_\kappa \qquad\qquad \kappa = 1,\dots,s \tag{4.22}$$

while the F_ρ ($\rho = 1,\dots,p$) are first class.

Now the matrix of the Poisson brackets of the second class constraints can be written

$$M = \begin{bmatrix} A & B \\ -\tilde{B} & 0 \end{bmatrix} \tag{4.23}$$

where

$$\left.\begin{aligned} A^\kappa_{\cdot h} &= \{\Theta_\kappa, \Theta_h\} \\ B^\kappa_{\ h} &= \{\Theta_\kappa, F_{p+h}\} \end{aligned}\right\} \quad (\kappa, h = 1,\dots,s) \tag{4.24}$$

The inverse of M (which will be useful in the following) is

$$M^{-1} = \begin{bmatrix} 0 & -\tilde{B}^{-1} \\ B^{-1} & B^{-1} A \tilde{B}^{-1} \end{bmatrix} \tag{4.25}$$

and

$$\det M = \det B \cdot \det(-\tilde{B}) \tag{4.26}$$

It follows that det $M \neq 0$ if and only if det $B \neq 0$. So the rank of the matrix $\| B^{k}_{.\mu} \|$ is equal to s .

Using this fact, the solution of the linear homogeneous system (4.20) can be found using the general solution given in the Appendix, with the choice $C^{\sigma}_{.\rho} = \delta^{\sigma}_{\rho}$.

It is

$$\bar{B}^{\kappa}_{.\rho} = \delta^{\kappa}_{\rho} - \delta^{\kappa}_{\kappa} (B^{-1})^{\kappa}_{h} \{ \Theta_h, F_\rho \} \tag{4.27}$$

where $\mu = 1 , \ldots, p+s$; $\rho = 1,\ldots,p$: $k,h = 1,\ldots,s$; and substituting in equation (4.11) we get

$$d\xi^{\alpha} = \left[\{ \xi^{\alpha}, F_\rho \} - \{ \xi^{\alpha}, F_{p+k} \} (B^{-1})^{\kappa}_{.h} \cdot \{ \Theta_h, F_\rho \} \right] d\omega^{\rho} \tag{4.28}$$

Let me observe that another choice of the solution (4.27), with $C^{\sigma}_{.\rho} \neq \delta^{\sigma}_{\rho}$, will have the meaning of a change in the parameters, from ω^{ρ} to some new $\bar{\omega}^{\rho}$ such that

$$C^{\sigma}_{.\rho} \cdot d\omega^{\rho} = d\bar{\omega}^{\sigma} \tag{4.29}$$

with

$$C^{\sigma}_{.\rho} = \frac{\partial \bar{\omega}^{\sigma}}{\partial \omega^{\rho}} \tag{4.30}$$

This means that the choice $C^{\sigma}_{\rho} = \delta^{\sigma}_{\rho}$ imply a determined choice of the parameters in eq. (4.28) .

Since by definition $\{F_{p+k}, F_\rho\} = 0$, the equation (4.28) can also be written in the form

$$\begin{aligned} d\xi^{\alpha} = \Big[& \{ \xi^{\alpha}, F_\rho \} - \{ \xi^{\alpha}, F_{p+k} \} (B^{-1})^{\kappa}_{.h} \{ \Theta_h, F_\rho \} + \\ & + \{ \xi^{\alpha}, \Theta_k \} (\tilde{B}^{-1})^{\kappa}_{.h} \{ F_{p+h}, F_\rho \} - \\ & - \{ \xi^{\alpha}, F_{p+k} \} (B^{-1})^{\kappa}_{.h} \{ \Theta_h, \Theta_m \} (\tilde{B}^{-1})^{m}_{.r} \{ F_{p+r}, F_\rho \} \Big] d\omega^{\rho} \end{aligned} \tag{4.31}$$

where in the r.h.s. we have added quantities which are zero.

We recongnise in (4.31) the structure of the Dirca brackets, with respect to the second class constraints F_{p+k} and Θ_k , $k=1,\ldots,s$. In writing (4.31)use was made of eq. (4.25) .

Finally we may write

$$d\xi^\alpha = \{\xi^\alpha, F_\rho\}^* \cdot d\omega^\rho \tag{4.32}$$

The integrability conditions for this set of equations (Hamilton-Dirac equations) can be easily verified by taking advantage of the properties of the Dirac's brackets. We know that the Dirac brackets verify both the Jacobi identity and the relation

$$\{f(\xi), g(\xi)\}^* = \frac{\partial f(\xi)}{\partial \xi^\alpha} \cdot \{\xi^\alpha, g(\xi)\}^*$$

This is enough to verify the integrability conditions, since

$$\frac{\partial^2 \xi^\alpha}{\partial\omega^\sigma \partial\omega^\rho} = \frac{\partial}{\partial\omega^\sigma}\{\xi^\alpha, F_\rho\}^* = \{\{\xi^\alpha, F_\rho\}^*, F_\sigma\}^* =$$

$$= \{\{\xi^\alpha, F_\sigma\}^*, F_\rho\}^* + \{\xi^\alpha, \{F_\rho, F_\sigma\}^*\}^*$$

but the last term in the r.h.s. is zero since, using eq. (4.28)

$$\{F_\rho, F_\sigma\}^* = \{F_\rho, F_\sigma\} - \{F_\rho, F_{\rho+k}\}(B^{-1})^k{}_h \{\Theta_k, F_\sigma\}$$

which vanishes due to the involutory character of the F_μ.

No particular hypothesis are necessary for the constraints Θ_k, except that they must be in number less than the F_μ, or better that $p \neq 0$, otherwise the only solution of (4.18) would be $d\theta^\mu = 0$.

Let me observe that the equations of motion (4.32) are weakly equal to the Hamilton equations (with the Poisson brackets in the place of the Dirac brackets). The result (4.32) is here meant to show that the procedure here developed is equivalent to that of Dirac. See also the following Section where the method is applied to the two-body problem.

5. Application of the Hamilton-Jacobi Method: the Two-Body Problem

Let me consider the Komar's constraints for the two-body system, where for simplicity we assume equal masses (see Ith lecture),

$$\begin{aligned} F_1 &= p^2 + 4q^2 - 4m^2 + 4\underset{\sim}{r}^2 \\ F_2 &= (p \cdot q) \end{aligned} \tag{5.1}$$

where

$$\underset{\sim}{r}^\mu = r^\mu - \frac{(r \cdot p)\, p^\mu}{p^2} \tag{5.2}$$

and

$$p = p_1 + p_2 \quad ; \quad x = (x_1 + x_2)/2$$
$$q = (p_2 - p_1)/2 \quad , \quad r = x_2 - x_1 \tag{5.3}$$

The constraints F_1 and F_2 are in involution, that is(*)

$$\{F_1, F_2\} = 0 \tag{5.4}$$

We have in this example: $\underline{n = 8}$, $\underline{p = 2}$.
Let us put

$$\left.\begin{aligned} \underline{q}_\lambda &= \epsilon^\mu_\lambda(p)\, q_\mu \quad , \quad (\lambda = 1,2,3) \\ \underline{r}_\lambda &= \epsilon^\mu_\lambda(q)\, r_\mu \\ \bar{q} &= (p \cdot q) \\ \bar{r} &= (p \cdot r) \end{aligned}\right\} \tag{5.5}$$

where

$$\epsilon^0_\lambda(p) = \frac{p^\lambda}{M} \quad ; \quad \epsilon^i_\lambda(p) = \delta_{i\lambda} + \frac{p_\lambda \cdot p_i}{M(p_0 + M)}$$
$$M = \sqrt{p^2} \tag{5.6}$$

The inverse relations are

$$q_\mu = -\epsilon_{\lambda,\mu}(p)\, \underline{q}_\lambda + \frac{\bar{q}}{p^2}\, p_\mu$$
$$r_\mu = -\epsilon_{\lambda,\mu}(p)\, \underline{r}_\lambda + \frac{\bar{r}}{p^2}\, p_\mu \tag{5.7}$$

The Poisson brackets of the new variables are

$$\{\underline{q}_\lambda, \underline{r}_{\lambda'}\} = -\delta_{\lambda\lambda'} \quad , \quad \{\bar{q}, \bar{r}\} = 1 \tag{5.8}$$

(*) Signature: +,--- ; $\{A, B\} = \frac{\partial A}{\partial p_i^\mu} \cdot \frac{\partial B}{\partial x^\mu_i} - \frac{\partial B}{\partial p_i^\mu} \cdot \frac{\partial A}{\partial x^\mu_i}$ so
$\{p^\mu, x^\nu\} = g^{\mu\nu}$; $\{q^\mu, r^\nu\} = g^{\mu\nu}$.

We may verify that the following objects are in involution with F_1 and F_2 and between themselves:

$$\left.\begin{aligned} F_3 &= \underline{q}_1^2 + c\,\underline{r}_1^2 - \epsilon_1 \\ F_4 &= \underline{q}_2^2 + c\,\underline{r}_2^2 - \epsilon_2 \\ F_5 &= \underline{q}_3^2 + c\,\underline{r}_3^2 - \epsilon_3 \end{aligned}\right\} \qquad (5.9)$$

$$\left.\begin{aligned} F_6 &= p_1 - k_1 \\ F_7 &= p_2 - k_2 \\ F_8 &= p_3 - k_3 \end{aligned}\right\} \qquad (5.9')$$

where ϵ_i, k_i (i = 1,2,3) are arbitrary constants.

The set (5.1), (5.9) and (5.9'), when we put $F_i = 0$ (i = 1,2,...,8), is equivalent to

$$\left.\begin{aligned} & p_0^2 - \vec{k}^2 - 4(\underline{q}_\lambda\,\underline{q}_\lambda + c\,\underline{r}_\lambda\,\underline{r}_\lambda) - 4m^2 = 0 \\ & k_0 q_0 - (\vec{k}\cdot\vec{q}) = 0 \\ & \underline{q}_i^2 + c\,\underline{r}_i^2 - \epsilon_i = 0 \quad , \quad i = 1,2,3 \\ & \vec{p} - \vec{k} = 0 \end{aligned}\right\} \qquad (5.10)$$

but where now it is

$$\underline{q}_\lambda = \epsilon_\lambda^\mu(k)\, q_\mu \quad , \quad \underline{r}_\lambda = \epsilon_\lambda^\mu(k)\, r_\mu \qquad (5.11)$$

and $$k^\mu \equiv (k^0, \vec{k})$$

where(≠)

$$k^0 = \sqrt{\vec{k}^2 + M^2} \quad , \quad M^2 = 4m^2 + 4(\epsilon_1 + \epsilon_2 + \epsilon_3) \qquad (5.12)$$

(≠) We have selected the positive value of k_0 for simplicity.

If we solve the constraints (5.10) in p^r and q^r (or in $\underline{q}_\lambda$ and $\bar{q}$), by selecting the positive values of the square roots, we get

$$p^r = k^r \qquad q_0 = (\vec{k}\cdot\vec{q})/k_0$$
$$\underline{q}_\lambda = \sqrt{\epsilon_\lambda - c\,\underline{r}_\lambda^2} \tag{5.13}$$

or, more explicity

$$\left.\begin{aligned} p_0 &= \sqrt{\vec{k}^2 + 4m^2 + 4(\epsilon_1+\epsilon_2+\epsilon_3)} \\ \vec{p} &= \vec{k} \\ q_0 &= \sum_{\lambda=1}^{3} \frac{1}{M}\, k_\lambda \sqrt{\epsilon_\lambda - c\,\underline{r}_\lambda^2} \\ q^i &= \sum_\lambda \epsilon_\lambda^i(k) \sqrt{\epsilon_\lambda - c\,\underline{r}_\lambda^2} \end{aligned}\right\} \tag{5.14}$$

where now all is expressed in terms of the six arbitrary constants ϵ_i and k_i.

If in (5.14) we eliminate these six constants, we recover the original constraints F_1 and F_2.

Equations (5.14) give the set (3.1) in this example. From equation (5.14) we may calculate the Hamilton-Jacobi function S:

$$\begin{aligned} dS &= -p_0\,dx_0 + \vec{p}\cdot d\vec{x} - q_0\,dr_0 + \vec{q}\cdot d\vec{r} = \\ &= -k^r dx_r + \sum_{\lambda=1}^{3} \epsilon_\lambda^r(k) \sqrt{\epsilon_\lambda - c\,\underline{r}_\lambda^2}\cdot dr_r \end{aligned} \tag{5.15}$$

(since from equations (5.14) we have $q^r = -\epsilon_\lambda^r(k)\sqrt{\epsilon_\lambda - c\,\underline{r}_\lambda^2}$.

We may express r_r in terms of $\underline{r}_\lambda$ and $\bar{r}$:

$$dr_r = -\epsilon_{\lambda r}(k)\, d\underline{r}_\lambda + \frac{k_r}{M^2}\, d\bar{r} \tag{5.16}$$

from which ($k^r \epsilon_{\lambda r}(k) = 0$, $\lambda = 1,2,3$)

$$-q^r dr_r = \sum_{\lambda=1}^{3} q_\lambda \cdot d\underline{r}_\lambda$$

so that dS can be written

$$dS = -k^r dx_r + \sum_\lambda \sqrt{\epsilon_\lambda - c\,\underline{r}_\lambda^2}\cdot d\underline{r}_\lambda \tag{5.17}$$

or

$$S = -(k\cdot x) + \frac{1}{2\sqrt{c}} \sum_{\lambda} \left[\sqrt{c}\, r_\lambda \sqrt{\varepsilon_\lambda - c\, r_\lambda^2} + \right.$$
$$\left. + \varepsilon_\lambda \cdot \arcsin \frac{\sqrt{c}\, r_\lambda}{\sqrt{\varepsilon_\lambda}} \right] + \text{const.} \qquad (5.18)$$

where S must be thought a function of x^r and r^r, and of the six constants ε_i, $\vec{k}$.

The solutions of the equations of motion can be got from S by putting equal to six new constants the derivatives of S with respect to ε_i and k_i :

$$\frac{\partial S}{\partial k_i} = h_i = -\frac{k_i}{k_0} x_0 + x_i + \sum_{\lambda} \sqrt{\varepsilon_\lambda - c\, r_\lambda^2} \cdot \frac{\partial r_\lambda}{\partial k_i} \qquad (5.19)$$

where

$$\frac{\partial r_\lambda}{\partial k_i} = \frac{1}{M(k_0+M)} \left[k_\lambda \cdot r_i - \delta_{\lambda i} (\vec{k}\cdot\vec{r}) \right] -$$
$$- \frac{\bar{r}}{M^2} \left(\delta_{\lambda i} - \frac{k_\lambda k_i}{k_0 (k_0+M)} \right) \qquad (5.20)$$

and

$$\frac{\partial S}{\partial \varepsilon_i} = a_i = -\frac{2x_0}{k_0} + \frac{2}{M^2} \sum_{\lambda} \frac{\bar{r}}{M k_0} k_\lambda \sqrt{\varepsilon_\lambda - c\, r_\lambda^2} +$$
$$+ \frac{1}{2\sqrt{c}} \arcsin \frac{\sqrt{c}\, r_i}{\sqrt{\varepsilon_i}} \qquad (5.21)$$

Equations (5.19) and (5.21) will give x_i and r_i in terms of x_0 and r_0 , and of the 12 constants (the correct number for the initial conditions for two particles) ε_i , k_i and h_i , a_i .

Following the considerations of Section 4, we may put the restriction $(p,r) = 0$ or $\bar{r} = 0$ on the solutions (5.19) and (5.21).

In this case we have the following simplication

$$a_i = -\frac{2x_0}{k_0} + \frac{1}{2\sqrt{c}} \arcsin \frac{\sqrt{c}\, r_i}{\sqrt{\varepsilon_i}} \qquad (5.22)$$

$$h_i = -\frac{k_i x_0}{k_0} + x_i + \sum_{\lambda} \sqrt{\varepsilon_\lambda - c\, r_\lambda^2} \cdot \frac{k_\lambda r_i - \delta_{\lambda i} (\vec{k}\cdot\vec{r})}{(M+k_0) M} \qquad (5.23)$$

where the last term in the r.h.s. of (5.23) is a constant

$$- \sum_{\lambda} \sqrt{\varepsilon_\lambda - c\, r_\lambda^2} \cdot \frac{k_\lambda r_i - \delta_{i\lambda} (\vec{k}\cdot\vec{r})}{M(k_0+M)} =$$

$$= \sum_\lambda \sqrt{\frac{\epsilon_i}{c}}\, k_\lambda \sqrt{\epsilon_\lambda} \cdot \sin\left[2\sqrt{c}\,(a_i - a_\lambda)\right]$$

We may redefine h_i and write

$$\left.\begin{aligned} x_i &= \bar{h}_i + \frac{k_i}{k_0}\, x_0 \\ r_i &= \sqrt{\frac{\epsilon_i}{c}}\, \sin\left[2\sqrt{c}\,\left(a_i + \frac{2x_0}{k_0}\right)\right] \end{aligned}\right\} \qquad (5.24)$$

from which finally we get the final form of the solutions

$$\left.\begin{aligned} x_i &= \bar{h}_i + \frac{k_i}{k_0}\, x_0 \\ r_i &= \sum_\lambda \left(\delta_{\lambda i} + \frac{k_\lambda k_i}{M(k_0+M)}\right) \cdot \sqrt{\frac{\epsilon_\lambda}{c}} \cdot \sin\left[2\sqrt{c}\,\left(a_\lambda + \frac{2x_0}{k_0}\right)\right] \\ r_0 &= \frac{1}{M} \sum_\lambda k_\lambda \sqrt{\frac{\epsilon_\lambda}{c}}\, \sin\left[2\sqrt{c}\,\left(a_\lambda + \frac{2x_0}{k_0}\right)\right] \end{aligned}\right\} \qquad (5.25)$$

For the momenta we have

$$\left.\begin{aligned} \vec{p} &= \vec{k} \qquad , \qquad p_0 = \sqrt{\vec{k}^2 + 4m^2 + 4 \sum_\lambda \epsilon_\lambda} \\ q_i &= \sum_\lambda \left(\delta_{\lambda i} + \frac{k_\lambda k_i}{M(k_0+M)}\right) \sqrt{\epsilon_\lambda} \cdot \cos\left[2\sqrt{c}\,\left(a_\lambda + \frac{2x_0}{k_0}\right)\right] \\ q_0 &= \sum_\lambda \frac{k_\lambda}{M} \sqrt{\epsilon_\lambda} \cdot \cos\left[2\sqrt{c}\,\left(a_\lambda + \frac{2x_0}{k_0}\right)\right] \end{aligned}\right\} \qquad (5.26)$$

It can be verified that these are the solutions which we could have got from the lagrangian equations of motion, after having eliminated the parameter τ in terms of x_0 .

Appendix

In order to demonstrate the equation (3.16), we will study the solutions of the system (3.14). Let us put

$$\xi^i_\rho = \frac{\partial x^i}{\partial \tau^\rho} \quad , \quad (i=1,\dots,n \; ; \; \rho = 1,\dots \rho \leq n) \tag{A.1}$$

$$M^a_{\cdot i} = \frac{\partial^2 \phi}{\partial c_a \partial x^i} \quad , \quad (a = p+1, \dots, n) \tag{A.2}$$

The rank of the matrix $M = \| M^a_{\cdot i} \|$ is by hypothesis $n-p$.

Let us use the following convention: a prime to some index will mean that it varies on the complementary set; for instance if $\alpha = 1,\dots,p$; $\alpha' = p+1,\dots,n$.

The system (3.14) can now be written

$$M^{\rho'}_{\cdot i} \, \xi^i_\rho = 0 \quad , \quad (\rho = 1,\dots,p \; ; \; \rho' = p+1,\dots n \; ; \; i = 1,\dots,n) \tag{A.3}$$

Without loss of generality we may suppose that

$$\left| M^{\rho'}_{\cdot \sigma'} \right| \neq 0 \tag{A.4}$$

so we can solve the system (A.3) in the $\xi^{\rho'}_\rho$ as functions of the ξ^σ_ρ which are left undetermined.

The solution is

$$\xi^{\rho'}_\rho = - (M^{-1})^{\rho'}_{\cdot \sigma'} \, M^{\sigma'}_{\cdot \sigma} \, \xi^\sigma_\rho \tag{A.5}$$

which can be written

$$\xi^i_\rho = \left(\delta^i_\sigma - \delta^i_{\rho'} (M^{-1})^{\rho'}_{\cdot \sigma'} \, M^{\sigma'}_{\cdot \sigma} \right) \cdot \xi^\sigma_\rho \tag{A.6}$$

since for $i = \sigma$ we have an identity.

In order to express the fact that the ξ^σ_ρ remain arbitrary, let us put them equal to the elements of an arbitrary matrix $C^\sigma_{\cdot \rho}$:

$$\xi^i_\rho = \left(\delta^i_\sigma - \delta^i_{\rho'} (M^{-1})^{\rho'}_{\cdot \sigma'} \cdot M^{\sigma'}_{\cdot \sigma} \right) \cdot C^\sigma_{\cdot \rho} \tag{A.7}$$

which for any $C^\sigma_{\cdot \rho}$ is solution of the system (A.3) .

If we have two solutions which differ for the choice of the $C^{\sigma}_{\cdot\rho}$:

$$\left.\begin{aligned} \xi^{i}_{\rho} &= \left(\delta^{i}_{\sigma} - \delta^{i}_{\rho'}\cdot(M^{-1})^{\rho'}_{\cdot\sigma'}\cdot M^{\sigma'}_{\cdot\sigma}\right) C_{1}{}^{\sigma}_{\cdot\rho} \\ \eta^{i}_{\rho} &= \left(\delta^{i}_{\sigma} - \delta^{i}_{\rho'}\cdot(M^{-1})^{\rho'}_{\cdot\sigma'}\cdot M^{\sigma'}_{\cdot\sigma}\right) C_{2}{}^{\sigma}_{\cdot\rho} \end{aligned}\right\} \qquad (A.8)$$

it exists a linear relation between ξ and η if one of the two matrices C_1, C_2 is not singular, say C_1 .

Indeed if we put

$$A_{\rho}{}^{\cdot\sigma}\cdot C_{1}{}^{\lambda}_{\cdot\sigma} = C_{2}{}^{\lambda}_{\cdot\rho} \qquad (A.9)$$

we get from (A.8)

$$\eta^{i}_{\rho} = A_{\rho}{}^{\cdot\sigma}\,\eta^{i}_{\sigma} \qquad (A.10)$$

Bibliographical Note

There is a huge literature on the Hamilton-Jacobi theory, so we will only give a short list of references intended as a guide for the reader:

Mathematical texbooks, where the theory of systems of partial differential equations of the first order can be found:

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CONSTRAINT RELATIVISTIC CANONICAL PARTICLE DYNAMICS

F. Rohrlich
Department of Physics
Syracuse University
Syracuse, NY 13210, USA

I. CLASSICAL THEORY

1. Introduction

Within the last few years very considerable progress has been made in the development of a theory of interacting point particles where the interaction is not described by an intermediate field but is direct. These lecture will summarize the present state of the art from the "Hamiltonian" point of view. Other formulations which are expected to be equivalent to this view (at least in a certain sense) include the Lagrangian formulation and the predictive dynamics. But neither of these seem to have been developed as far as the (generalized) Hamiltonian formulation. They are described by other contributors to this conference.

The five words of the title of my lectures are meant in the following sense.

Dynamics: This theory is not meant to be a fundamental theory of interactions and is not intended as an alternative to quantum field theory. Rather, it is a dynamics in the sense that it describes the motion of particles under given "forces" which are to a large extent arbitrary, although important restrictions on them will be seen to emerge from the requirements of invariance, consistency, cluster property, etc.

Particle: This is a direct interaction theory between particles. Fields as mediators of interactions are not considered although they are not excluded in an implicit sense: if the fields of a field theoretic interaction are eliminated in favor of the particles an example of the direct interaction theory may result. Also, direct interaction theory may be an approximation to a field theory. This is of special importance for phenomenological applications.

Canonical: The starting point of the theory includes a canonical algebra (a symplectic space). But it is not assumed that these canoni-

cal variables have direct physical meaning. In general, physical variables emerge at a later stage of the theory.

Let the N particles be labeled by a,b,... and the components of the canonical variables q and p by k, ℓ, ... then the canonical algebra is in classical physics

$$\{ q_{fa}^{k}, p_{b\ell} \} = \delta_{ab} \cdot \delta_{\ell}^{k} \tag{1}$$

i.e. a Poisson bracket algebra. In quantum physics it is a commutator algebra,

$$[q_{fa}^{k}, p_{b\ell}] = i \, \delta_{ab} \cdot \delta_{\ell}^{k} \tag{2}$$

More generally, one envisions a 2Nd dimensional symplectic space Γ where d is the number of independent components of q (or p).

Relativistic: The notion of special relativity implies two concepts of relevance here, Poincaré invariance and the irreducible representations associated with free particles. Poincaré invariance requires that the functional form of the generators of the Poincaré transformations $P^{\mu}(q,p)$ and $M^{\mu\nu}(q,p)$ implies the Poincaré algebra as a consequence of the q, p algebra. It also requires the physical description (for example the classical world lines of the interacting particles) to be covariant under Poincaré transformations. We shall here develop a manifestly covariant theory although manifest covariance is not necessary for a relativistic theory. Specifically, we have, using μ, ν to take on the values 0,1,2,3,

$$P_{\mu} = \sum_{a} p_{a\mu} \quad , \qquad M^{\mu\nu} = \sum_{a} (q_a \wedge p_a)^{\mu\nu} \tag{3}$$

The irreducible representations associated with a unique mass $m > 0$ and spin $s \geq 0$, $[m,s]$ is characterized by the values of the two Casimir invariants

$$p^{\mu} p_{\mu} = - m^2 \tag{4}$$

and

$$w^{\mu} w_{\mu} = + m^2 s(s+1) \tag{5}$$

where

$$w^{\mu} = S_{*}^{\mu\nu} p_{\nu} \tag{6}$$

is the Pauli-Lubanski vector and $S^{*\mu\nu}$ is the dual to the spin tensor $S^{\mu\nu}$. Equation (4) is the mass shell equation of a free particle and will be a basic starting point of the theory.

Constraints: A covariant formulation does not use the minimum number of variables (the six components of $\vec{q}_a$ and $\vec{p}_a$ for particle a corresponding to the six degrees of freedom of a spinless particle) but uses four vectors q_a^μ and $p_{a\mu}$ thus adding two variables, q_a^o and p_{ao} for each particle. These additional $2N$ variables must eventually be eliminated. Suitable constraints are necessary for this purpose so that one finds that the need for constraints goes hand in hand with a manifestly covariant formulation.

Hamiltonian dynamics with constraints was developed by Dirac, Bergmann, Komar, Sudarshan, Mukunda, and others. Some references are provided at the end of these lectures.

2. Constraint Theory

In the time available here it would not be possible to give even an introduction into constraint theory[1,4]. However, since some basic notions will be needed in the following presentation I shall give here a few definitions.

Constraints are functions of the q and p which vanish in some part of the phase space Γ. They cannot vanish everywhere in Γ because that would contradict the canonical algebra which required all q,p to be independent from one another. The symbol $\approx$ (weak equality) is used to indicate validity of an equation in only part of Γ. Constraints thus have the form

$$C_m(q,p) \approx 0 \tag{7}$$

First class constraints are characterized by the fact that they commute with one another (i.e. have vanishing Poisson brackets). Since each first class constraint permits the elimination of 2 variables, one needs only $C=N$ such constraints to eliminate $2N$ variables.

Second class constraints do not commute with one another and in fact must satisfy the following condition: $2S$ second class constraints (they always occur in even numbers) define a $2S \times 2S$ matrix D,

$$D = (D_{mn}) \,, \qquad D_{mn} = \{C_m, C_n\} \tag{8}$$

with the property

$$|D| = \det D \neq 0 \tag{9}$$

Every set of constraints can be divided into first and second class constraints. For the elimination of $2N$ variables one needs $F+2S$ constraints such that $F+S = N$.

$2S$ second class constraints reduce the $8N$ dimensional phase space $\Gamma(8N)$ to a $8N-2S$ dimensional one, Γ'. On Γ' the variable q,p satisfy an algebra with a different bracket, the Dirac Bracket. That bracket, $\{\cdot,\cdot\}^*$, is defined by

$$\{A,B\}^* = \{A,B\} - \{A,C_m\}\,(D^{-1})_{mn}\{C_n,B\} \tag{10}$$

where the summation convention is assumed.

Given any variable A one can associate to it a variable A^* such that

$$A^* \approx A \tag{11}$$

which commutes (i.e. has vanishing Poisson bracket) with all the $2S$ second class constraints

$$\{A^*, C_m\} \approx 0 \qquad (m = 1, 2, \ldots, 2S) \tag{12}$$

These "star variables" are obtained as follows[2]:

$$A^* = A - \{A, C_m\}\,(D^{-1})_{mn}\,C_n \tag{13}$$

The q^* and p^* span a symplectic space isomorphic to Γ'. Their Poisson brackets on Γ' are just the Dirac brackets. In fact,

$$\{A^*, B^*\} \approx \{A^*, B\} \approx \{A, B^*\} \approx \{A,B\}^* \tag{14}$$

If $S=N$, i.e. if all constraints are second class then the q^* and p^* are the <u>physical</u> variables since they obey the constraints. For example, the physical position $x_a^\mu = q_a^{\mu *} \approx q_a^\mu$ is equal to the position on the subspace Γ'. That subspace is (in that case of $S=N$) the $6N$ dimensional physical phase space which we shall denote by Φ^*.

3. <u>The Geometric Structure</u>

The theory in the form in which we shall present it here developed out of the independent proposals by Arens[5], Todorov[6], and Komar[7], as well as a number of others (see the references for a few representative papers).

Starting with the free mass shells (4) of N particles,

$$p_a^2 + m_a^2 \approx 0 \qquad (a = 1, 2, \ldots, N) \tag{15}$$

we see that these are N first class constraints, i.e. the correct number for the elimination of 2N variables. Interaction is introduced by making the mass depend on the variables. More precisely, (15) is replaced by

$$K_a \equiv p_a^2 + m_a^2 + \phi_a(q,p) \approx 0 \qquad (a=1,2,\ldots,N) \tag{16}$$

where ϕ_a vanishes in the free particle limit and is in general a function of all the q_b and p_b $(b=1,2,\ldots,N)$. It is called the interaction function.

One now assumes that the first class character of (15) is preserved in (16),

$$\{K_a, K_b\} \approx 0 \qquad (a, b = 1, 2, \ldots, N) \tag{17}$$

Sazdjian has given arguments for the necessity of (17) which we shall not repeat here.

The N equations (16) restrict $\Gamma(8N)$ to a subspace, $\mathcal{M}(7N)$, the general mass shell hypersuface. Consider any point $\gamma = (q_1, q_2, \ldots q_N, p_1, \ldots p_N) \in \mathcal{M}$. The quantity K_a can be used as a generator of a trajectory through γ :

$$\frac{d\gamma}{d\tau_a} = \{\gamma, K_a\} \tag{18}$$

The parameter along that trajectory is here denoted by τ_a. The integrability condition of the equations (18) is just the first class condition (17). It ensures that the N trajectories generated by the N K_a all lie in $\mathcal{M}$; they span an N-dimensional surface $\Sigma(N)$. Since every $\gamma \in \mathcal{M}$ lies on some (unique) surface Σ, it follows that (18) generated a foliation of $\mathcal{M}$,

$$\mathcal{M} = \Sigma \otimes \Phi \tag{19}$$

Or, one can define a quotient space[9] Φ by

$$\Phi = \mathcal{M}/\Sigma \tag{20}$$

Φ is the 6N dimensional phase space of physical points: all points on a given leaf Σ are physically equivalent since they differ only

by weakly vanishing terms. The physical motion must be a canonical transformation of Φ into itself, a canonical automorphism.

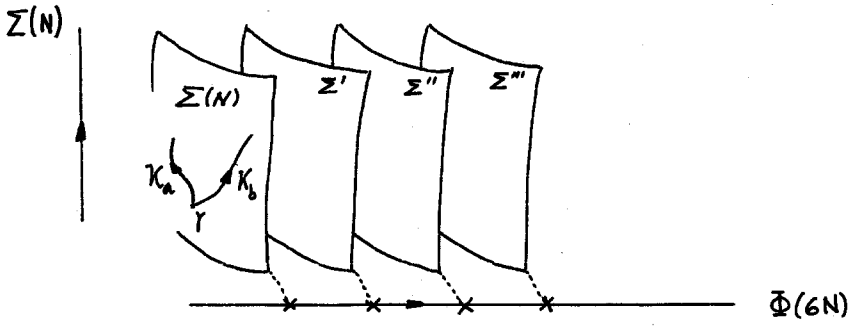

Fig. 1. The general mass shell hypersurface $\mathcal{M}(7N)$ as a foliation $\Sigma(N) \bullet\!\!- \Phi(6N)$. Those trajectories from γ on Σ are drawn that are generated by K_a and by K_b.

If one chooses a set of functions

$$\omega_a = \omega_a(q,p) \tag{21}$$

and constructs the linear combination

$$H = \sum_a \omega_a K_a \tag{22}$$

one obtains a generator that also generates a trajectory from γ on Σ. For every set $\{\omega_a\}$ a different trajectory will result. These trajectories on Σ are called gauge motion since they are clearly not physical motions and are generated by constraints. The points on a gauge motion trajectory all correspond to the same point on Φ, since all points of a given Σ correspond to the same point on Φ.

4. Fixations

The geometrical picture leads to the physical phase space Φ as a quotient space, equation (20). But it does not provide equations of motion. For that purpose some "time parametrization" needs to be introduced which permits an explicit description of the evolution of the system[3,9]. In Minkowski space such a parametrization amounts to a specification of a family of three-dimensional hypersurfaces (usually taken to be spacelike) labelled by some invariant time parameter τ.

For the N particle system such a time parametrization amounts to a specification of the variables $q_a{}^o$ in terms of the other variables in phase space. Thus, one needs N equations (necessarily weak ones) involving the N $q_a{}^o$. Since these are to fix the hypersurfaces which characterize successive states of the system they are called "fixations",

$$\chi_a(q,p,\tau) \approx 0 \qquad (a=1,2,\ldots,N) \tag{23}$$

This family of surfaces is labelled by the monotonically increasing parameter τ, and is required to intersect the family of surfaces (16), i.e. one requires

$$|\Delta| \equiv \det \Delta \neq 0 \tag{24}$$

$$\Delta_{ab} \equiv \{\chi_a, K_b\} \tag{25}$$

One can think of (23) as providing a functional dependence of the $q_a{}^o$ on τ and on the other variables.

From the point of view of constraint theory the two sets of equations (16) and (23) form a set of 2N second class constraints, equation (24) corresponding to (9). Such a set has the property that the coefficient functions ω_a of (22) are uniquely determined by them if one requires these constraints to be conserved under the evolution generator H.

Because of (17) the K_a are trivially conserved,

$$\{K_a, H\} \approx 0 \tag{26}$$

For the χ_a one has, because of the explicit τ dependence

$$\frac{\partial \chi_a}{\partial \tau} + \{\chi_a, H\} \approx 0 \tag{27}$$

or

$$\omega_a = -(\Delta^{-1})_{ab} \frac{\partial \chi_b}{\partial \tau} \qquad \text{(summation convention)} \tag{28}$$

Thus, the fixations (23) fix the evolution generator uniquely to be

$$H = -K_a (\Delta^{-1})_{ab} \frac{\partial \chi_b}{\partial \tau} \tag{29}$$

This generator yields a unique trajectory

$$\frac{d\gamma}{d\tau} = \{\gamma, H\} \approx \{\gamma, \kappa_a\} \cdot (\Delta^{-1})_{ab} \cdot \frac{\partial \chi_b}{\partial \tau} \tag{30}$$

Since γ has the 2N components q_a^μ, $p_{b\nu}$, this equation is just a compact form of Hamilton's equations of motion for our system.

From (18) we learn that the N times τ_a are related to τ by

$$\frac{d\tau_a}{d\tau} = \omega_a = -(\Delta^{-1})_{ab} \cdot \frac{\partial \chi_b}{\partial \tau} \tag{31}$$

This equation relates the "many-time" formulation to the "single-time" formulation of dynamics. That relation is unique for any choice of the fixations χ_a, (23).

However, it is at this point difficult to understand why (30) should have anything to do with the physical motion of the particles since (18) is clearly a gauge motion and any linear combinations of gauge motions is also a gauge motion: the point on $\underline{\Phi}$ does not move because the trajectory remains in a given leaf.

5. Physical Variables[12]

The physical variables $\gamma^* = q_1^{\mu *}, \ldots q_N^{\mu *}, p_{1\mu}^*, \ldots p_{N\mu}^*$ are by definition variables that equal γ on $\underline{\Phi}$ and that commute (have vanishing Poisson brackets) on $\underline{\Phi}$ with all the constraints. When all constraints are second class these are just the star variables constructed in (13). The 2N constraints (7) consist in the present case of the N constraints(16) and the N fixations (23). The matrix D has the form

$$D = \begin{pmatrix} \theta & \Delta \\ -\tilde{\Delta} & 0 \end{pmatrix} \tag{32}$$

where

$$\theta_{ab} \equiv \{\chi_a, \chi_b\} \tag{33}$$

and its inverse is

$$D^{-1} = \begin{pmatrix} 0 & -\tilde{\Delta}^{-1} \\ \Delta^{-1} & \Delta^{-1}\theta\tilde{\Delta}^{-1} \end{pmatrix} \tag{34}$$

Substitution into (13) then leads to the physical variables

$$\gamma^* = \gamma - \{\gamma, \chi_a\}\,(-\tilde{\Delta}^{-1})_{ab} K_b - \{\gamma, K_a\}\,[\,(\Delta^{-1})_{ab}\chi_b + (\Delta^{-1}\theta\tilde{\Delta}^{-1})_{ab} K_b] \tag{35}$$

One verifies easily that they satisfy

$$\gamma^* \approx \gamma\,, \qquad \{\gamma^*, K_a\} \approx 0\,, \qquad \{\gamma^*, \chi_a\} \approx 0 \tag{36}$$

The equations of motion of these variables are therefore

$$\frac{d\gamma^*}{d\tau} = \frac{\partial \gamma^*}{\partial \tau} + \{\gamma^*, H\} \approx \frac{\partial \gamma^*}{\partial \tau} \tag{37}$$

since they depend on τ explicitly via the χ_a . But γ and the K_a are not explicit functions of τ so that (37) applied to (35) gives

$$\frac{d\gamma^*}{d\tau} \approx -\{\gamma, K_a\}\,(\Delta^{-1})_{ab}\,\frac{\partial \chi_a}{\partial \tau} \tag{38}$$

These then are the equations of motion of the physical variables. And we now discover that these are just the Hamiltonian equations (30) on the physical subspace.

One concludes that the gauge motion associated with a particular set of fixations is just the physical motion on Φ . For a fixed value of τ the γ^* span a physical space Φ^* which is a <u>different</u> representation of the quotient space Φ for different τ .

We now turn to a brief discussion of the center-of-momentum variables (CM variables[10]). This is a choice of variables often advocated in the literature. It differs significantly from the above individual particle variables (IP variables). Instead of 8N IP variables one has $8N + 8$ CM variables $Q^\mu, P^\mu, \xi_a^\mu, \pi_a^\mu$ which must satisfy an algebra which is a realization of the Poincaré algebra when

$$M^{\mu\nu} = (Q \wedge P)^{\mu\nu} + \sum_a (\xi_a \wedge \pi_a)^{\mu\nu} \tag{39}$$

The most convenient choice is the covariant canonical realization

$$\{Q^\mu, P_\nu\} = \delta^\mu_\nu\,, \qquad \{\xi_a^\mu, \pi_{b\nu}\} = \delta_{ab}\cdot\delta^\mu_\nu \tag{40}$$

with all other Poisson brackets vanishing. Other realizations imply relations between the CM variables so that not all $8N + 8$ of them are independent dynamical variables. (See the appendix of my paper

in Nuclear Physics B112,177(1976). An example of that is the case where one chooses for Q the Newton-Wigner position variables.

The meaning of the CM variables can be deduced from (39): Q and P are the CM position and momentum of the system. $\mathfrak{z}_a$ and π_a are internal (relative) positions and momenta. (39) is the separation of the generalized angular momentum into one of the system as a whole and one which is the sum of the individual internal particle angular momenta. Particle spin is here ignored.

In terms of these variables there is a certain arbitrariness in choosing constraints since the mass shell is no longer accessible. A much more serious problem, however, is the fact that these variables lead to difficulties when N particle systems (N >2) are considered in which the particles interact by means of separable interactions. It does not seem possible to satisfy the cluster decomposition property in terms of the CM variables. We shall return to this property later.

On the other hand, the CM variables are very satisfactory and intuitively desirable for bound state problems of two-body systems. Also N-body systems can be treated this way if no scattering states are possible (nonseparable interactions). The relation between the CM variables and the IP variables has been studied in several papers (see e.g. M.J. King and F. Rohrlich).

6. Many-body Forces

So far no attention was paid to the first class condition (17) of the constraints. In view of the form of the K_a, (16), this is really a condition on the interaction functions ϕ_a: only those interaction functions are permitted for which (17) holds. But what are these functions?

For N=2 the answer to this questions is very simple. Since one can here always choose $\phi_1 = \phi_2 = \phi$ the equation

$$\{p_1^2 + \phi\,,\; p_2^2 + \phi\} = 0 \tag{41}$$

leads to $p_1 \frac{\partial \phi}{\partial q_1} - p_2 \frac{\partial \phi}{\partial q_2} = 0$ or, because of translation invariance of ϕ, i.e. $\frac{\partial \phi}{\partial q_1} = -\frac{\partial \phi}{\partial q_2}$ one finds

$$P \cdot \frac{\partial \phi}{\partial q_{12}} = 0 \qquad (P = p_1 + p_2\,;\; q_{12} = q_1 - q_2) \tag{42}$$

This result states that the two-body interaction function can depend

on q_1 and q_2 only through

$$q_{12}^{\perp\mu} = q_{12\alpha} P_\perp^{\alpha\mu} \tag{43}$$

where $P_\perp^{\mu\nu}$ is the projection orthogonal to the total momentum P^μ,

$$P_\perp^{\mu\nu} = \eta^{\mu\nu} + \hat{P}^\mu \hat{P}^\nu \tag{44}$$

Here $\eta^{\mu\nu}$ is the Minkowski metric and $\hat{P}^\mu$ is the unit vector $P^\mu/\sqrt{-P^2}$. (We use trace $\eta = +2$). The functions ϕ are therefore scalar functions constructed from the three fourvectors $q_{12}^{\perp\mu}$, p_1^μ p_2^ν.

For N=3 the problem is already much more difficult. One must solve the three equations

$$\{ p_a^2 + \phi_{ab} + \phi_{ac},\ p_b^2 + \phi_{bc} + \phi_{ba} \} = 0 \qquad (a\ b\ c = \text{cyclic } 123) \tag{45}$$

for the three functions ϕ_{ab}, with $\phi_{ab} = \phi_{ba}$. And now one finds that there is no nontrivial solution to this problem![11]

Therefore, it is necessary to extend the type of interaction functions considered. One can no longer limit oneself to two-body interactions. For N=3 three-body interactions must be present for any nontrivial system.

This is a feature characteristic to the formalism based on generalized mass shells as first class constraints.

An explicit solution for such three-body forces was obtained by H. Sazdjian[10]. He also found a solution for the N-body system which requires n-body forces with $n = 2,3,4,\ldots N$. Earlier work on this problem was done by S.N. Sokolov[13].

Since Dr. Sazdjian will speak on his work at this conference no further discussion of it is necessary here.

The necessity of many-body forces in direct interaction dynamics should not be surprising. As early as 1939 Primakoff and Holstein[14] showed that when retarded interactions of fields are replaced by direct interactions many-body forces arise. It is the price one pays for restricting oneself to direct interactions.

7. The Cluster Decomposition[11]

Any particle dynamics which is to be physically meaningful must satisfy the cluster property when separable interactions are involved.

A particle is subject to separable forces if these forces cease in

the limit as that particle is removed to infinite spacelike distance relative to all the other particles. Such forces in general admit bound states as well as scattering states of the system.

If the particles of an N particle system interact with one another by separable forces the interaction functions ϕ_a must vanish asymptotically as the relative spacelike distances approach infinity. If the system separates into two clusters of N' and N'' particles, $N' + N'' = N$, the cluster decomposition property requires that in the limit

(1) the dynamics of each cluster is independent of the dynamics of the other cluster,

(2) the dynamics of each cluster is independent of its history,

(3) the cluster decomposition is Poincaré invariant.

For $\phi_a = \sum_{n=2}^{N} \phi_a^{(n)}$ separability requires, as particle $\underline{a}$ separates,

$$\lim_{|q_{ab}|\to\infty} \phi_a = 0 \tag{46a}$$

$$\lim_{|q_{ab}|\to\infty} \phi_b \quad \text{independent of } \underline{a} \,. \tag{46b}$$

The phase space Γ separates into two non-intersecting spaces Γ' and Γ'' such that

$$\Gamma \longrightarrow \Gamma' \oplus \Gamma'' \tag{47}$$

and the Poincaré generators of the two clusters becomes additive and commutative, $\{M^{\mu\nu'}, M^{\rho\sigma''}\} = 0$, etc. The evolution operator then also separates

$$H = H' + H'' \tag{48}$$

where

$$H' = \sum_{1}^{N'} \omega_a' K_a' \,, \qquad H'' = \sum_{1}^{N''} \omega_a'' K_a'' \tag{49}$$

In order to satisfy requirement (2) it is necessary that H' and H'' be independent of P^μ the total momentum of the parent system of N particles. If one uses CM variables this requirement seems very difficult to satisfy.

Now the ω_a of the original (parent) system as well as the ω_a' and ω_a'' must be Poincaré invariant and in particular translation invariant. The χ_a however, even if chosen Lorentz invariant cannot be translation invariant. It follows from (28), therefore, that both $\partial\chi_a/\partial\tau$ and

Δ must nevertheless be translation invariant. This is a necessary condition to satisfy (3) .

In order to satisfy (1) it is necessary that the ϕ_a depend only on spacelike relative distances. Only those are affected by the limit in (46). On the other hand, the fixations must depend on timelike position vectors. While the ϕ_a depend only on $q_{ab}^{\perp}$ where $\perp$ indicates orthogonality to some timelike vector (e.g. a timelike momentum), the χ_a must depend on the $q_a^{//}$ where // means a component of q_a parallel to some timelike vector. Otherwise the fixations do not permit elimination of the coordinate times $q_a^{\,o}$.

The limits involved in the cluster separation affect only the $q_{ab}^{\perp}$ and does not take the $q_{ab}^{//}$ to infinity. If particle b goes to infinity ω_a would not become independent of b if it is a function of $q_{ab}^{//}$ unless ω_a also depends on $q_{ab}^{\perp}$ and in such a way that the $q_{ab}^{//}$ dependence is eliminated as $|q_{ab}^{\perp}| \longrightarrow \infty$. One sees therefore that the fixations χ_a must be so chosen that the resultant ω_a (using (28)) will be consistent with the cluster requirements. As an example one observes that the very reasonable looking fixations $\chi_a \equiv q_a \cdot \hat{P} + \tau_a = 0$ do not yield separable ω_a for $N > 2$.

8. Interactions

The above dynamics is based on the free mass shell and its modification due to interaction. The free mass shell

$$\eta^{\mu\nu} p_\mu p_\nu + m^2 = 0 \tag{50}$$

can be modified in three different ways. The best known of these is the gauge interaction

$$p_\mu \longrightarrow p_\mu - g A_\mu$$

leading to

$$\eta^{\mu\nu}(p_\mu - g A_\mu)(p_\nu - g A_\nu) + m^2 = 0 \tag{51}$$

The gauge functions A_μ can be group valued, e.g. of the form

$$A_\mu = \sum_i A_\mu^i t_i \quad , \qquad [t_i, t_j] = C_{ij}^{k} t_k$$

for a Lie group (non-abelian gauge interaction).

A second way of modifying (50) is by generalizing Minkowski space to a Riemann space of indefinite metric $\eta^{\mu\nu} \longrightarrow g^{\mu\nu}$ yielding

$$g^{\mu\nu} p_\mu p_\nu + m^2 = 0 \qquad (52)$$

Physically this means that the particle is in a gravitational background field.

A third modification is the generalization of the mass term to a function of the dynamical variables

$$\eta^{\mu\nu} p_\mu p_\nu + M^2(q,p) = 0 \,, \qquad M^2 = m^2 + \phi(q,p) \qquad (53)$$

which is the form of direct interaction we used in (16).

In principle, both modifications (51) and (52) can be written in the form (53) but the inverse is not true in general.

The relation to a gauge field theory is seen by taking for A_μ a field, $A_\mu(x)$, and specifying field equations for it that relate it to the particles as sources. The solution of the field equations then expresses A_μ as a functional of the particle variables which may be very complicated and is in general nonlocal in space and time. The question of a reduction to the form (53) then becomes a non-trivial problem.

Concluding Remarks to I.

The classical relativistic constraint dynamics of particle systems has made tremendous progress in the last few years. It is now on fairly sound foundations and the essential features seem to be understood. This includes in particular the cluster problem and the status of many-body forces.

The main questions that are still not well understood are primarily the following:

(1) The freedom available in choosing three-body interactions when the two-body interactions are known, is not yet fully clarified. The same holds for n-body forces with $n > 3$.

(2) The description of spin for classical particles has not yet been worked out.

(3) The global structure of the theory is not yet understood. To this end a fully coordinate independent formulation must be provided (fiber bundle language).

(4) The relation of this canonical theory to other formulations (Lagrangian, predictive dynamics) has not yet been sufficiently clarified.

But it is not too soon to begin applying the theory to various specific problems where classical particle dynamics is a valid description.

II. QUANTUM THEORY

1. Introduction

The canonical quantization of classical relativistic constraint dynamics with direct interaction leads to a relativistic quantum dynamics of directly interacting particles. This theory is intermediate between nonrelativistic quantum mechanics and relativistic quantum field theory: it is a relativistic quantum theory with a finite number of degrees of freedom. Is there a consistent theory of this nature?

Knowing the classical theory one anticipates various difficulties of which the following are representative:

(1) In a covariant formulation q_a^μ and $p_{a\mu}$ will be operator valued four-vectors. This means that in addition to the well-known operators $\vec{q}_a$ and $\vec{p}_a$ we shall also have an operator of energy $p_a{}^o$ and an operator of time $q_a{}^o$. The latter would imply an uncertainty relation $\Delta E \cdot \Delta t$ which is independent of the position-momentum uncertainty relation. A host of questions of interpretation and measurement theory arise here.

(2) The interaction operator $\phi(q,p)$ becomes in the Schrödinger representation $\phi(q, i\frac{d}{dq})$ which leads to a Schrödinger equation of higher than second order, and in general to a quasi-differential equation.

(3) Covariance required a Hilbert space $\mathcal{H}$ of state vectors, $\mathcal{H} = L^2(R^{4N})$. What is the physical interpretation of this?

(4) The constraints which are weak equations in the classical theory become equations of the form

$$K_a|\psi\rangle = 0$$

Since K_a has a continuous spectrum $|\psi\rangle$ cannot be in $\mathcal{H}$. One must use wave packets or generalize to a rigged Hilbert space. A similar problem, but less acute is encountered in ordinary quantum mechanics.

Our attempt at a relativistic constraint quantum dynamics is less than a year old although some important earlier work is related to it and helped shape our thinking on this matter. In any case we are far from having answers to all the above questions. Nevertheless, important progress has been made and we shall summarize a good part of it in the following pages.

2. Quantization

The heuristic process called "quantization" is essentially an educa-

ted guess at a quantum theory whose classical limit is the given classical theory. There is no mathematical rigor in that process. Only the limit is to be well defined mathematically. Quantization is an art.

Attempts by mathematicians to make quantization into a rigorous procedure (geometric quantization) leads to a theory which has little to do with the quantum mechanics of the physicist.

In canonical quantization[20] one replaces the Poisson bracket algebra of the fundamental variables by a commutor algebra

$$[q_a^\mu, p_{b\nu}] = i\,\delta_{ab}\,\delta^\mu_\nu \tag{54}$$

with q_a^μ, $p_{b\nu}$ self adjoint operators on $\mathcal{H} = L^2(R^{4N})$. The fundamental dynamical equation is the generation of a trajectory by a given K_a. The latter is also a self adjoint operator now,

$$K_a \equiv p_a^2 + m_a^2 + \Phi_a(q,p) \tag{55}$$

with Φ_a being (a suitably ordered) function of the q and p. This dynamical equation thus reads

$$i\frac{d\gamma^H}{d\tau_a} = [\gamma^H, K_a^H] \tag{56}$$

with γ^H the set of $8N$ operators $q^H_{1\mu}, \ldots, p^H_{N\mu}$. The index H indicates that we are here in the Heisenberg picture in which all the time dependence is in the operators, the state vectors $|\psi_H\rangle$ being time independent. The time here is a set of N parameters τ_a one for each particle. Thus, we have a many-time theory, corresponding to the classical equation (18). A single time formalism would require the ω_a, (31). However, they are not necessary in a scattering theory where only the asymptotic free states are needed.

It is conceivable that (56) cannot hold everywhere in $\mathcal{H}$ and we shall be content to have it hold on a set of states $|\psi_H\rangle$ that span a subspace $\Phi \subset \mathcal{H}$. That space would then be the physical Hilbert space,

$$i\frac{d\gamma^H}{d\tau_a}|\psi_H\rangle = [\gamma^H, K_a^H]|\psi_H\rangle \tag{57}$$

The Schrödinger picture is characterized by time independent operators γ^S. All τ_a dependence is in the state vectors $|\psi_S\rangle$. The transformation $U([\tau])$, $[\tau] = \tau_1, \ldots, \tau_N$ maps one picture on the other,

$$|\psi_S\rangle = U([\tau])|\psi_H\rangle \tag{58}$$

and one finds by standard methods that U must satisfy

$$i\frac{dU}{d\tau_a} = K_a^s U \tag{59}$$

Therefore the $|\psi_s\rangle$ must satisfy

$$i\frac{d}{d\tau_a}|\psi_s\rangle = K_a^s|\psi_s\rangle \tag{60}$$

which is a set of N simultaneous differential equations for $|\psi_s\rangle = |\psi_s(\tau_1,\ldots,\tau_N)\rangle$. Its integrability conditions are

$$[K_a^s, K_b^s]|\psi_s\rangle = 0 \tag{61}$$

in which one recognizes the quantum analog of the first class constraint condition (17).

The solution of (59) is

$$U([\tau]) = \prod_{a=1}^{N} e^{-i K_a^s \tau_a} \tag{62}$$

At this point the technical assumption of stability of Φ under the operators K_a is necessary.

Equations (60) and (61) are the fundamental equations of the relativistic constraint quantum dynamics. In fact, eqs. (60) can be regarded as the relativistic generalizations of the Schrödinger equations. We see that they are in general not of second order, as predicted in (2) . But one notices that the difficulty anticipated in (4) did not arise: the time derivative term permits one to have solutions $|\psi_s\rangle \in \Phi$. Physically, this means that the quantum mechanical mass shell is <u>not</u> sharp but has a finite (though presumably very small) width. This situation arises naturally in the quantization process and has not been put in by hand.

One can show that the fundamental equations lead to a classical limit consistent with the theory developed in I . In particular the matrix elements of K_a^s are constraints which vanish in the classical limit reproducing the classical constraints.

3. <u>Many-Time Quantum Dynamics</u>

The relativistic Schrödinger equation (60) is not entirely new to the physics literature. As a single τ formulation it was proposed by Stueckelberg[15], and was later discussed by Feynman[16], Schwinger[17], and others.

The work I am reporting is based on joint papers with L. Horwitz[20,21] who had recently studied (60) in the single time formulation[18,19]. Independently Droz-Vincent[22] is carrying out similar studies in a rigged Hilbert space.

The difficult set of differential equations is expected to be solvable at least in a perturbation expansion. Following the well-known techniques of quantum field theory it is thus convenient to transform into the Dirac picture

$$|\psi\rangle = U_0^{-1} |\psi_s\rangle \tag{63}$$

$$U_0([\tau]) = \prod_{a=1}^{N} e^{-i K_a^0 \tau_a} \tag{64}$$

where

$$K_a^0 = p_a^2 + m_a^2 \tag{65}$$

is the unperturbed mass shell in the Schrödinger picture. One verifies that the Dirac picture operators γ^D satisfy the free particle equations. Therefore the momenta $p_a^D = p_a^S$ are $[\tau]$-independent and

$$k_a^0 \equiv K_a^{0\,D} = U_0^{-1} K_a^0 U_0 \tag{66}$$

The integrability condition (61) becomes

$$[k_a, k_b]|\psi\rangle = 0 \tag{67}$$

The fundamental equations in the Dirac picture are the U_0 transforms of (60),

$$i \frac{d}{d\tau} |\psi\rangle = \phi_a |\psi\rangle \tag{68}$$

where $\phi_a \equiv \Phi_a^D$ is the interaction operator in the Dirac picture. Their solution can be written as

$$|\psi[\tau]\rangle = U([\tau],[\sigma])|\psi[\sigma]\rangle \tag{69}$$

with $U([\tau],[\sigma])$ a unitary operator obtained from the ϕ_a by integration,

$$U([\tau],[\sigma]) = U_{[\tau]}(\tau_1,\sigma_1) \cdot U_{\sigma_1,\tau_2\ldots\tau_N}(\tau_2,\sigma_2) \cdot \ldots \cdot U_{\sigma_1\ldots\sigma_{N-1},\tau_N}(\tau_N,\sigma_N) \tag{70}$$

A typical factor in this product is

$$\mathcal{U}_{q\ldots\sigma_{i-1},\tau_i\ldots\tau_N}(\tau_i,\sigma_i) = \left(e^{-i\int_{\sigma_i}^{\tau_i} \phi_i(\sigma_1,\ldots\sigma_{i-1},\tau_i',\tau_{i+1},\ldots,\tau_N)\,d\tau_i'} \right)_+ \qquad (71)$$

and the symbol $(\ )_+$ indicates positive τ-ordering increasing from right to left.

The integrability condition (67) permits one to prove that on $|\psi\rangle$ the order in which the $[\sigma']$ are integrated to $[\tau]$ is arbitrary. It follows that the path from $[\sigma]$ to $[\tau]$ in N-dimensional τ-space is arbitrary, yielding the same $\mathcal{U}([\tau],[\sigma])$ for any path.

The operator $\mathcal{U}([\tau],[\sigma])$ permits one to compute a (generalized) Feynman propagator in this many-time theory. Let $\psi(q,[\tau]) = \langle q|\psi\rangle$ then

$$\psi(q,[\tau]) = \int \langle q|\mathcal{U}([\tau],[\sigma])|q'\rangle\, dq'.\ \psi(q',[\sigma]) \qquad (72)$$

with

$$G(q,[\tau];q',[\sigma]) = \langle q|\mathcal{U}([\tau],[\sigma])|q'\rangle \qquad (73)$$

the associated propagator.

These expressions and the associated space-time picture are a good starting point for the discussion of the physical interpretation of the theory and for an understanding of the space $L^2(R^{4N})$. We shall not discuss it here.

4. Scattering Theory

In the Dirac picture one has the asymptotic states

$$\lim_{[\tau]\to-\infty} |\psi\rangle = |\psi_{in}\rangle \qquad (74)\ \text{in}$$

$$\lim_{[\tau]\to+\infty} |\psi\rangle = |\psi_{out}\rangle \qquad (74)\ \text{out}$$

The Schrödinger states then have the limits

$$\lim_{[\tau]\to-\infty} \| |\psi_s\rangle - \mathcal{U}_0[\tau]|\psi_{in}\rangle \| = 0$$

and similarly for $[\tau] \to +\infty$.

The two operators (62) and (64) permit one to define the wave operators (Møller operators),

$$\lim_{[\tau]\to\mp\infty} U^{-1}([\tau])\, U_0([\tau]) = \Omega_\pm \tag{75}$$

which relate the Heisenberg states to the in- and out- states,

$$|\psi_H\rangle = \Omega_+ |\psi_{in}\rangle = \Omega_- |\psi_{out}\rangle \tag{76}$$

As before, one can prove that the order of the N limits in (75) is arbitrary. The existence of the limits depends on the interaction functions. A sufficient condition for the existence is

$$\int_{-\infty}^{T} \left\| \Phi_a^S \cdot \prod_{b=1}^{N} e^{-iK_b^0 \tau_b} |\psi_{in}\rangle \right\| \cdot d\tau_a < \infty \quad , \quad \forall a \tag{77}$$

Finally one defines the S-operator in the conventional way,

$$S = \Omega_-^+ \cdot \Omega_+ \tag{78}$$

and one finds

$$|\psi_{out}\rangle = S |\psi_{in}\rangle \tag{79}$$

In order to check on the cluster decomposition property one needs the translation operators

$$U_a(s_a) = e^{ip_a \cdot s_a} \qquad (s_a^\mu \text{ spacelike}) \ . \tag{80}$$

The interaction is separable if

$$\lim_{s_a^2\to\infty} U_a(s_a)\, \Phi_a^S\, U_a^{-1}(s_a) = 0 \tag{81}$$

$$\lim_{s_b^2\to\infty} U_b(s_b)\, \Phi_a^S\, U_b^{-1}(s_b) = \Phi_a' \quad , \quad (a \neq b) \tag{82}$$

where Φ_a' is independent of b .

By means of these operators one can then prove that if two clusters C' and C'' separate, the S operator factors in the limit,

$$S |\psi_{in}\rangle \longrightarrow S' \cdot S'' |\psi_{in}\rangle \tag{83}$$

An important result in scattering theory is the generalization of the Lippmann-Schwinger equation to the present theory. It takes two alternative forms. One uses the free mass shell in the denominator

and reads for the two-body case (without assuming $\Phi_1 = \Phi_2$)[21]

$$W_+^\dagger \Psi^{(+)}_{H k_1 k_2} = \psi_{in\, k_1 k_2} - \frac{1}{K_1^0 - (k_1^2 + m_1^2) - i\epsilon_1} \cdot \frac{1}{K_2^0 - (k_2^2 + m_2) - i\epsilon_2} V \Psi^{(+)}_{H k_1 k_2} \qquad (84)$$

Here $\psi_{in\, k_1 k_2} = \langle k_1 k_2 | \psi_{in} \rangle$ is the incident wave function and $\Psi^{(+)}_{H k_1 k_2} = \langle k_1 k_2 | \Psi_H \rangle^+$ is the Heisenberg wave function involving outgoing waves. The role of the conventional potential is here played by the operator V (Schrödinger picture)

$$V = \Phi_1^S \Phi_2^S + [\Phi_1^S, K_2^0] \qquad (85)$$

The occurrence of the operator W_+ in (84) has its origin in the possibility that each particle a can evolve due to its interacting mass shell operator $K_a \neq K_a^0$ without explicitly affecting the other one,

$$W_+ \equiv 1 + (\Omega_+^{(1)} - 1) + (\Omega_+^{(2)} - 1) \qquad (86)$$

$$\Omega_+^{(a)} = \lim_{\tau_a \to -\infty} e^{i K_a \tau_a} \cdot e^{-i K_a^0 \tau_a} \qquad (86)$$

Equation (84) can be made the starting point of a perturbation expansion in $V \cdot W_+^{\dagger -1}$.

Concluding Remarks to II.

There is increasing evidence that a consistent relativistic constraint quantum dynamics of N directly interacting particles does exist. We are still far from understanding this theory conceptually but the mathematical framework seems to have fallen into place. It now appears that this level of theory lies intermediate between (non-relativistic) quantum mechanics and (relativistic) quantum field theory. That means a relativistic quantum theory with a finite number of degrees of freedom seems possible.

In addition to the conceptual questions various others are still left to be studied. In particular, one notes the following:

(1) The theory is to be generalized to include particle spin, especially spin $\frac{1}{2}$. Some attempts in this direction have already been made.

(2) The quantum mechanical analog of the classical relation between three-body and two-body interactions needs to be understood (and similarly for n-body interactions); we have not yet explicit solutions for interactions Φ_a that satisfy equation (61).

(3) It is not clear how this theory applies to electrodynamics. How

can quantum electrodynamics be approximated by a theory with a finite number of degrees of freedom?[24]

(4) A similar question relates to quantum field theory in general. Can pair production as well as single particle production be described?

(5) There exists a relativistic scattering theory for multi-channel processes (see the lecture by F. Coester) how is that theory related to the constraint theory developed here? Can constraint quantum dynamics be developed into a fully general multi-channel theory?

Finally, as for the classical theory it is not too early to begin applying relativistic constraint dynamics to various realistic problems in the quantum domain. An excellent beginning has already beem made by the recent work of Van Alstine and Crater[23] on the quarkonium systems. More applications like that are needed. They will help establish relativistic constraint dynamics as an important new field of study.

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CONSTRAINT HAMILTONIAN MECHANICS OF DIRECTLY INTERACTING RELATIVISTIC PARTICLES

I.T. Todorov[*]
Fakultät für Physik, Universität Bielefeld
D-4800 Bielefedld 1

Introduction

The relativistic dynamics of a single particle in an external field was created -by Poincaré and Planck- in the early days of the special theory of relativity. The theory of a finite system of interacting relativistic particles -apart from some scattered and inconclusive attempts in the past- is only taking shape in recent years. Not only the road to it has been plagued with difficulties, but the very legitimacy of the problem has been questioned on the ground that a finite system of particles interacting "at-a-distance" (without an intermediary field) is violating the "Nahewirkungsprinzip" which is thought to be inherent to the theory of relativity.

If we look back to the sixties and early seventies, the situation with the relativistic 2-body problem -to single out one specific topic- appears rather queer. On one hand all existing allegedly field theoretic fine structure and Lamb shift calculations use (on top of quantum electrodynamics) some 2-particle equation (most efficiently, a Hamiltonian type quasipotential equation). On the other hand, a so-called "no interaction theorem" was put forward (and proved), stating that a relativistic canonical Hamiltonian formalism is only consistent with a free particle motion. (It brings to memory the ancient anecdote about Zeno proving the impossibility of motion and his opponent just walking in answer.)

Now we not only pretend to be moving, but we are also ready to explain why we are able to do so.

The present notes (which can be regarded as a concise version of a more comprehensive text in preparation -see T5)) purport to give an overall view on the constraint Hamiltonian approach to the subject

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[*] Permanent address: Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia 1184, Bulgaria.

starting with the classical relativistic mechanics of a single particle in an external field and ending with a derivation of the fine structure spectrum of a 2-particle bound state in quantum electrodynamics.

The reader can get an idea about the organization of material from the table of contents.

A note on the list of references

Although the subject matter of these notes has never been truly fashionable the number of publication in the field is depressingly large. (We are aware of a few hundreds of articles, a list of which will appear in T5).) The references listed at the end of these notes represent just a small section of that huge body and we have had no trustworthy criterion in making such a selection.

The present note aims to supply a short guide to the list of references (in order to give us the freedom not to interrupt too often the systematic exposition with historical and bibliographical comments).

There have been several lines of development in relativistic particle mechanics which now happily converge to a coherent overall picture.

The early stage of development of the subject can be traced back from references E1), D1,2), P5), W2), H2) (see also later work V2) in these lines as well as the review by Professor Hill at this Workshop). Two unrelated developments in the sixties have left a lasting imprint on the subject. One is the "no-interaction theorem" of Currie, Jordan, Sudarshan (CJS) C4), Leutwyler L3), and Hill H3) (see also K1), M2)). The other is the development of the quasi-potential approach by Logunov, Tavkhelidze and others L6,7) (for later reviews, see F2), R3)).

The difficult problem of constructing separable relativistic N-particle interactions for $N \geq 3$ has a long history (which is reviewed in R6), and S3) as well as in Professor Coester's lecture at this Workshop). The necessity for manybody forces has been realized already in the thirties (see P4)). A semirelativistic approximation scheme for evaluating 3-particle interactions, which have to accompany the sum of 2-body potentials in order to make the theory consistent, was proposed by Foldy F3) (see also P2)). Landmarks in the solution of the corresponding quantum mechanical problem (using Möller's wave operators) were set by Coester C1) and Sokolov S4,5). Iterative schemes (in terms of powers of the 2-particle interaction and their derivatives) were developed recently in S3) and B6) (see also T5)). In our presentation in Sec. 6 we follow the work of Bidikov and the author B6).

The development of classical relativistic mechanics in the seventies has gone in three major lines which nowadays are merging together.

The starting point of the predictive relativistic mechanics of Bel and others B3-5), F5), L1,2) is the picture of retarded particle interaction via a classical (say, electromagnetic) field which is then substituted by an equivalent Hamiltonian picture.

The second line, characterized by the use of a canonical phase space Hamiltonian formalism has its origin in two sets of papers by Dirac D2,3). It gave rise to several related developments including the CJS work mentioned above, the work of Arens A2) and Droz-Vincent D5), as well as the constraint Hamiltonian approach started in T4) and reviewed in these lectures. This latter work is based on two sources: Dirac's "Generalized Hamiltonian dynamics" D3) and its further development (see F1), H1)) and the local version of the quasipotential approach developed in T3) and R3,4). (An attempt to construct classical relativistic mechanics using inspiration from the corresponding quantum theory was made earlier in F4).) For later work in this direction, see e.g. R5), K2)+, C2,3), M3), L8), H4), S3), T5).

The third parallel line consists in the development of the singular Lagrangian approach (see T1), D4), G1,3), L5) as well as references to the earlier work of the Japanese school cited in the first of these papers). In practical terms (as far as explicit examples go) this approach seems to be equivalent to a special case of the constraint Hamiltonian approach in which the interaction is independent of the energy (or, more precisely, it may only depend on the total momentum P through the orthogonal relative distance r (given by Eq. (5.16) below)).

Our treatment of gauge dependence of canonical world lines and gauge invariance of asymptotic results (Sec. 7) follows the work M3) of Molotkov and the author. A similar result was obtained in the Lagrangian framework in G1).

The space-time formulation of relativistic particle dynamics in terms of second order differential systems follows recent work by Nikolov (see N1)).

Recent developments of relativistic Hamiltonian (and Lagrangian) mechanics can also be found in G2), L4), P1,3).

+) Unfortunately, it has to be noted that the lack of originality in the first two papers in the series K2) is only matched by the lack of references there to the author's predecessors.

1. Space-time formulation of relativistic particle mechanics

A world line in a space-time manifold M is a 1-dimensional (time-like) submanifold of M, usually given by parametric equations of the type

$$x^\mu = x^\mu(t) \qquad \mu = 0,1,2,3 \quad ; \quad -\infty < t < +\infty \,.$$

For a Galilean invariant (non-relativistic) system there is a privileged choice of the evolution parameter t, namely, the time-component x^0 of the 4-vector x. It does not change under homogeneous Galilean transformations (and, in general, only the origin of the time axis may be shifted). For a relativistic system this is not the case: the separation between space and time components of x depends on the choice of the Lorentz frame (and the proper time, which is a natural evolution parameter for a single /massive/ particle, has no universal extension to many particle systems). This makes it desirable to have a formulation of relativistic dynamics which does not depend on the choice of t (provided, say, that $\dot{x}^0 = \frac{dx^0}{dt} > 0$). (Such a "physical motivation" should be superfluous for a geometer: by definition, a world line is a parametrization independent object, and a theory designed to determine it (from some "initial data") should not depend in an essential way on the evolution parameter either.)

In a Newton like formulation already the choice of initial data poses a problem, if we insist on reparametrization invariance. Indeed, the momentary state of a particle is conventionally given by its position and velocity at a given time. However, the 4-velocity depends on the choice of the evolution parameter: for t replaced by $f(t)$ ($\dot{f} = \frac{df}{dt} > 0$) $\dot{x}$ goes into $\dot{f}\dot{x}$. The fact that under an arbitrary (monotonous) change of t the 4-velocity is just dilated (by a positive factor) suggests a simple way out of this difficulty. The initial data should consist of a space-time point x and a future pointing tangent ray to the world line at that point. This leads us to (a slight modification of) the mathematical notion of a (first order) differential system (which assigns to each point $x \in M$ a k-dimensional subspace in the tangent space $T_x M$). In order to be able to formulate a parametrization independent version of Newton's second law, an extension of this concept is needed to second order systems. Such an extension has been worked out in the present context by P. Nikolov (see N1)).

In an attempt to give an idea about this approach without entering the differential geometric subtleties inherent to it, we shall consider

in detail the simplest case of a single particle system in an external field.

Let M be a pseudo-Riemannian manifold of signature $-+++$ (whose curvature may be physically interpreted as a manifestation of an external gravitational field). The space of 4-velocities at a point x is the future cone in the tangent space T_xM consisting of (non-zero) 4-vectors $\dot{x}$ such that $\dot{x}^0 \geq |\underline{\dot{x}}|$. The union of all these spaces (for x varying in M), which can be regarded as a subbundle of the tangent bundle TM, will be denoted by $T_>M$. The parametrization independent concept of velocity is given by the ray $[\dot{x}]$ of all vectors of the type $\lambda\dot{x}$ where $\dot{x}$ is fixed and $\lambda > 0$. The bundle of all such rays is the projectivization $P(T_>M)$ of $T_>M$. Thus we shall identify the space of (instantaneous) states of a spinless point particle with the 7-dimensional manifold $P(T_>M)$ whose points in local coordinates are given by the pairs $(x, [\dot{x}])$. (The notation $[\dot{x}]$ means that $\dot{x}^\mu$ are used as homogeneous coordinates in $P(T_{x>}M)$).

The manifold $P(T_>M)$ may, alternatively, be viewed as the space of initial conditions for a (1-particle) mechanical system. In order to introduce a parametrization independent concept of acceleration and to have room for Newtonian type of equations of motion, we have just to repeat the above construction taking the tangent space $T\,P(T_>M)$ and its projectivization, the 13-dimensional space $P(T\,P(T_>M))$. Let $\pi : P(T_>M) \to M$ be the projection which makes correspond to every point $(x, [\dot{x}])$ of the fibre bundle $P(T_>M)$ the point x of the base space M, and let π^T be the corresponding tangent map

$$\pi^T : T_y P(T_>M) \longrightarrow T_{\pi(y)} M .$$

Denote by $\tilde{\pi}$ the projective counterpart of π^T ($\tilde{\pi}$ maps $P(T\,P(T_>M))$ into $P(T_>M)$). A (1-dimensional) second order differential system is defined as a section σ: $P(T_>M) \longrightarrow P(T\,P(T_>M))$ satisfying the condition

$$\tilde{\pi}(\sigma(y)) = y \quad \text{for every} \quad y \in P(T_>M). \tag{1.1}$$

We shall demonstrate that this geometric concept provides a parametrization independent generalization of Newton's equations of motion. To this end we first introduce independent (rather than homogeneous) coordinates u^i in velocity space. We shall assume, for the sake of definiteness, that we are dealing with a massive particle with a time like world line. In this case it is convenient to use the normalized 4-velocity

$$u^\mu = \frac{\dot{x}^\mu}{\sqrt{-\dot{x}^2}} \qquad (u^2 \equiv u_\mu u^\mu = -1 \;;\; u_\mu \equiv g_{\mu\nu} u^\nu) \tag{1.2}$$

whose space components u^i can serve as independent coordinates in $P(T_x > M)$. (More generally, if a light like motion is also allowed, one can take instead $v^i = \frac{\dot{x}^i}{\dot{x}^0}$ as independent parameters.) With this notation we can choose $(x^\mu, u^i; [\dot{x}^\mu, \dot{u}^i])$ as local parameters in $P(TP(T > M))$ (regarding $\dot{x}^\mu, u^i$ as homogeneous coordinates in the 6-dimensional fibre). A second order differential system has a local representation

$$\sigma : (x^\mu, u^i) \longrightarrow (x^\mu, u^i ; [S^\nu(x,u), F^j(x,u)]),$$

Condition (1.1) now implies that S^ν should be proportional to the 4-velocity:

$$S^\nu(x,u) = \lambda(x,u)\, \dot{x}^\nu .$$

Let $\tilde{\ell}$ be an integral curve of σ in $P(T > M)$, so that

$$T_{(x,u)}\tilde{\ell} = \sigma(x,u) \quad \text{for all} \quad (x,u) \in \tilde{\ell} . \tag{1.3}$$

If we introduce at this point an evolution parameter τ on $\tilde{\ell}$, then

$$\left[\frac{dx^\mu}{d\tau}, \frac{du^i}{d\tau}\right] = [\lambda \dot{x}^\mu, F^i] . \tag{1.4}$$

Eq. (1.4) implies the Newton-like equation

$$m(\tau) \cdot \frac{du^i}{d\tau} = F^i . \tag{1.5}$$

The multiplier $m(\tau)$ depends (in general) on the choice of evolution parameter and on the definition of the force F. Eq. (1.5) can also written in a covariant 4-dimensional form

$$m \cdot \frac{du^\mu}{d\tau} = F^\mu \qquad (\text{for} \quad u^2 = g_{\mu\nu} u^\mu u^\nu = -1) \tag{1.6}$$

if we define F^0 from the orthogonality relation

$$uF \equiv g_{\mu\nu} u^\mu F^\nu = 0 \tag{1.7}$$

(which is a consistency condition for u being a normalized 4-velocity).

We can produce manifestly reparametrization invariant equations of type (1.6) starting from an action principle with a Lagrangian which is a homogeneous function of degree 1 in $\dot{x}$. An example of physical importance is provided by the Lagrangian for a massive charged particle in an external gravitational and electromagnetic field:

$$L = -m\sqrt{-g_{\mu\nu}(x) \cdot \dot{x}^\mu \dot{x}^\nu} + e\, A_\mu(x) \cdot \dot{x}^\mu \tag{1.8}$$

The Euler-Lagrange equations of motion can in this case be written in the form

$$m\, g_{\mu\nu}(x) \cdot \frac{Du^\mu}{D\tau} = e\, F_{\mu\nu} \dot{x}^\nu \tag{1.9}$$

where u is given by (1.2), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $\frac{D u^\mu}{D\tau}$ is the covariant derivative:

$$\frac{Du^\mu}{D\tau} = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\lambda\nu} u^\lambda \dot{x}^\nu \quad , \quad \Gamma^\mu_{\lambda\nu} = \frac{1}{2} g^{\mu\rho} \left(\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\lambda\rho} - \partial_\rho g_{\lambda\nu} \right) \tag{1.10}$$

We shall end up this section by writing down without further explanation the relevant definitions in the N-particle case.

A relativistic N-particle dynamics is defined by the commutative diagram

$$\left.\begin{array}{ccc} \left(P(TP(T_>M))\right)^N & \xrightarrow{\tilde{\pi}} & \left(P(T_>M)\right)^N \\ \sigma \uparrow \downarrow & & \downarrow \pi \\ \left(P(T_>M)\right)^N & \xrightarrow{\pi} & M^N \end{array}\right\} \quad \tilde{\pi} \cdot \sigma = 1 \tag{1.11}$$

where σ is an involutive section. (A section σ is called involutive if for any pair of vector fields X_1, X_2 the condition $X_a(x) \in \sigma$ $(a=1,2)$ implies $[X_1, X_2] \in \sigma$. According to Frobenious theorem the differential system σ is integrable iff it is involutive. Eq. (1.11) includes the requirement that σ lies in the domain of the "projective tangent map" $\pi_.$) The restriction σ_k of the differential system σ to $P(T M_k)$ satisfies all conditions of a 1-particle dynamics. The "external force" F_k, acting on the k-th particle, will, in general, appear as a function of the coordinates and velocities of all particles.

We now proceed to the discussion of symmetry. Let G be a transformation group of M whose induced action $(V_g, \tilde{V}_g; g \in G)$ on $(P(T_>M), P(T\,P(T_>M)))$ is a bundle homomorphism; in other words, if $\tilde{\pi}$ is the projection in the fibre bundle $P(T\,P(T_>M)) \xrightarrow{\tilde{\pi}} P(T_>M)$, then $\tilde{\pi} \cdot V_g = V_g \cdot \tilde{\pi}$.

We say that G is a symmetry group of σ (regarded as a section in the above fibre bundle) if

$$\tilde{V}_g \circ \sigma \circ V_g^{-1}(y) = \sigma(y) \,, \qquad g \in G \,, \; y \in P(T_>M) \,. \tag{1.12}$$

The notion extends in an obvious way to the N-particle case.

The largest symmetry of M which naturally appears in our framework is the conformal group locally isomorphic to $SO_0(D,2)$. The largest symmetry group for a particle with a fixed positive mass is the Poincaré group whose connected component will be denoted by $\mathcal{P}_+^\uparrow$.

We shall illustrate here the implication of Poincaré invariance on the example of a 1-particle system in D-dimensional space-time, assuming the standard (affine) action of $\mathcal{P}_+^\uparrow$ on M for which

$$V(a,\Lambda) : (x, [\dot{x}]) \longrightarrow (\Lambda x + a, [\Lambda \dot{x}]) \,. \tag{1.13}$$

We shall consider the $(2D-1)$ dimensional open submanifold $P(T_+M)$ of $P(T_>M)$ defined as the set of pairs $(x, [\dot{x}])$ for which $\dot{x}$ is (positive) timelike. It is a homogeneous space of $\mathcal{P}_+^\uparrow$ with stability

group of a point isomorphic to $SO(D-1)$.

For space-time dimension $D>2$ this "little group" acts nontrivially on the accelerations and hence on the forces leaving no non-vanishing $(D-1)$-vector invariant. Consequently, the condition of Poincaré invariance of the 1-particle dynamics leads to free motion ($F=0$) . For $D=2$, however, the little group $SO(1)$ is itself trivial and imposes no extra condition on the force so that the above result does not hold. Indeed, there exists a non-trivial Poincaré invariant 1-particle dynamics in 2-dimensional space-time. Using the proper time variable we can write the $\mathcal{P}_+^\uparrow$-invariant equation

$$\dot{u}_\mu = k\,\varepsilon_{\mu\nu}\,u^\nu \quad \text{where} \quad u^2+1 \equiv u^\mu u_\mu + 1 = 0 \;; \quad \varepsilon_{10} = -\varepsilon_{01}\,(=\varepsilon^{01}) = 1 \,. \tag{1.14}$$

Its solution satisfying the initial condition $v \equiv (u^1/u^0)\big|_{\tau=0} = \mathrm{th}\,\alpha$ is $u^0 = \mathrm{ch}\,(\alpha + k\tau)$, $u^1 = \mathrm{sh}\,(\alpha + k\tau)$. The particle world line is in this case a branch of a hyperbola with isotropic asymptots.

Note, however, that the system (1.14) is not invariant under space reflections. There is in fact no non-trivial (smooth) 1-particle dynamical system, invariant under the orthochronous Poincaré group $\mathcal{P}^\uparrow$. (The condition of smoothness is important, since otherwise the system governed by the equation

$$\dot{u}_\mu = k\,\varepsilon_{\mu\nu}\cdot \mathrm{sign}\,\dot{u}_1 \cdot u^\nu \qquad , \quad k>0$$

provides an example of a $\mathcal{P}^\uparrow$-invariant system with nonstraight world lines for $D=2$).

Going back to higher space-time dimensions (including the realistic case $D=4$) we conjecture that there exist nontrivial Poincaré invariant N-particle dynamics for $N \geqslant 2$ since the stability subgroup of almost all points of $(P(T_>M))^N$ is trivial.
(This conjecture will be further justified in Sec. 7C .)

We omit here the discussion of spin (a 1-spinning-particle system in an external field is recently studied in Vl)[+]); the phase space of a free classical spinning particle is described in Sec. 3 .

+) Note that the authors of Vl) use the term "chronometric invariance" for reparametrization invariance.

2. Hamiltonian constraint for a charged spinless particle in an external field

The idea behind the constraint Hamiltonian approach to relativistic particle dynamics is best illustrated by the simple example of a charged particle in an external field.

We start with an auxiliary 8-dimensional "large phase space" $\Gamma = T^*M$ where M is a (4-dimensional pseudo-Riemannian manifold of signature (-+++) whose metric tensor can be regarded as describing an external gravitational field. The cotangent bundle[+)] Γ has a natural symplectic structure, given by the 2-form

$$\omega = dx^\mu \wedge dp_\mu \tag{2.1}$$

or, equivalently by the canonical Poisson bracket relations

$$\{x^\mu, x^\nu\} = 0 = \{p_\mu, p_\nu\} \quad ; \quad \{x^\mu, p_\nu\} = \delta^\mu_\nu . \tag{2.2}$$

It turns out that the constraint which allows to express the particle energy $E = -P_0$ as a function of its 3-momentum and the external field also determines the equations of motion. We can identify the (generalized) Hamiltonian with the constraint

$$H = H(p, A; \lambda) = \frac{\lambda}{2}\left[m^2 + g^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu)\right] \approx 0 \quad (\lambda > 0) \tag{2.3}$$

The weak equality sign ($\approx$) indicates that, in evaluating Poisson brackets, x and p should be regarded as independent variables and the constraint (2.3) should only be applied after performing all differentiations. The positive factor λ, which is allowed to depend on the point in phase space, plays the role of a Lagrange multiplier and is related to the choice of evolution (or "time") parameter. Indeed, the Hamiltonian equations of motion for x,

$$\dot{x}^\mu = \{x^\mu, H\} = \frac{\partial H}{\partial p_\mu} \approx \lambda g^{\mu\nu}(p_\nu - eA_\nu), \tag{2.4}$$

show that a change in λ is equivalent to a change in the time scale. For a positive mass m it follows from (2.3) (2.4) that λ is proportional to the invariant length of the 4-velocity:

+) A concise and readable exposition of the basic prerequisites of symplectic geometry is contained in some 75 pages of the excellent treatise by Treves T6). An even shorter summary (of what is needed for reading the present notes) is given in the first chapter of T5).

$$\dot{x}^2 \equiv g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\lambda^2 m^2 \qquad (2.5)$$

The reparametrization invariance of space-time observables (such as the particle world line or the 7-dimensional instantaneous state of a particle, introduced in the preceding section) is expressed in the Hamiltonian picture as their independence of the Lagrange multiplier λ .

Note that the constraint (2.3) was introduced by Dirac D1) in the early forties; he was the first to realize that a constraint of this type not only excludes extra variables but also defines the dynamics.

The λ independence of observables indicates that the physics of the problem is completely characterized by giving a (7-dimensional) surface

$$\mathcal{M} \equiv \mathcal{M}_1(m, A(x)) =$$

$$= \{(x,p) \in T^*M;\ m^2 + g^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu) = 0,\quad p^0 - eA^0 > 0\} \qquad (2.6)$$

in Γ , the <u>generalized 1-particle mass-shell</u>. The inequality $p^0 - eA^0 > 0$ in (2.6) is necessary and sufficient in order to have a positive time-like or light-like velocity $\dot{x}$ (for $m \geq 0$, $\lambda > 0$, as assumed).

A straightforward way to establish a correspondence between the space time picture of Sec. 1 and the present constraint Hamiltonian description of a 1-particle system is provided by the Legendre transformation

$$H(p,x)\ (\approx 0) \longrightarrow L(x,\dot{x}) = p\dot{x} - H(p,x) \qquad (2.7)$$

where p is regarded as a function of the positions and velocities, defined (implicitely) by the Hamiltonian equation $\dot{x} = \frac{\partial H}{\partial p}$. For H given by (2.3) we find

$$p_\mu = e A_\mu(x) + \frac{1}{\lambda} g_{\mu\nu}(x)\cdot\dot{x}^\nu \qquad (2.8a)$$

$$L = \frac{1}{2\lambda} g_{\mu\nu}(x)\,\dot{x}^\mu\dot{x}^\nu - \frac{\lambda}{2} m^2 + e\dot{x}^\mu A_\mu(x) \qquad (2.8b)$$

For a positive mass particle the Lagrange multiplier λ can be excluded from the constraint equation $\frac{\partial L}{\partial \lambda} = 0$ (which coincides with (2.5)); the resulting Lagrangian

$$L = -m\sqrt{-\dot{x}^2} + e\dot{x}A \qquad (2.9)$$

coincides with Eq. (1.9) of the preceding section (and leads to the

same reparametrization invariant equations of motion).

Remark. There is a freedom in the definition of the nonconserved interacting particle momentum and the corresponding symplectic structure. We have chosen to describe the interaction of a charged particle with an external electromagnetic field by "deforming" the mass shell. The same physics can be described by a deformation of the canonical Poisson bracket structure (keeping the mass shell intact). Indeed, introducing the generalized momentum $\pi = p - eA$, we can rewrite the Hamiltonian constraint (2.3) in an A-independent form,

$$H = \frac{1}{2}\left(m^2 + g^{\mu\nu}\pi_\mu\pi_\nu\right) \approx 0 . \tag{2.10}$$

The interaction with the electromagnetic field then reappears in the Poisson brackets among the generalized momenta:

$$\{\pi_\mu, \pi_\nu\} = -e\left(\{p_\mu, A_\nu\} + \{A_\mu, p_\nu\}\right) = eF_{\mu\nu} \qquad \left(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\right). \tag{2.11}$$

It corresponds to the symplectic form

$$\omega = dx^\mu \wedge d\pi_\mu - \frac{e}{2} F_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu . \tag{2.12}$$

We leave it to the reader to verify that the Hamiltonian (2.10) and the Poisson brackets (2.11) lead to the same equations of motion (1.10) (for the x's) as before.

3. The phase space of a classical spinning particle

The large phase space Γ_s^+ of a classical spinning particle will be identified with a 10 dimensional symplectic submanifold of the 14 dimensional vector space $\mathcal{J}_{14}$ spanned by the space-time coordinates x^μ and the Poincaré generators p_λ and $J_{\mu\nu}$ which satisfy the standard (canonical) Poisson bracket relations

$$\{x^\mu, x^\nu\} = 0 = \{p_\mu, p_\nu\} \quad ; \quad \{x^\mu, p_\nu\} = \delta^\mu_\nu \tag{3.1a}$$

$$\{x^\lambda, J_{\mu\nu}\} = \delta^\lambda_\nu x_\mu - \delta^\lambda_\mu x_\nu \quad ; \quad \{J_{\mu\nu}, p_\lambda\} = \eta_{\mu\lambda} p_\nu - \eta_{\nu\lambda} p_\mu \tag{3.1b}$$

$$\{J_{\varkappa\lambda}, J_{\mu\nu}\} = \eta_{\varkappa\mu} J_{\lambda\nu} - \eta_{\varkappa\nu} J_{\lambda\mu} - \eta_{\lambda\mu} J_{\varkappa\nu} + \eta_{\lambda\nu} J_{\varkappa\mu} . \tag{3.1c}$$

We shall define the mass-spin shell $\mathcal{M}_{ms} \subset \Gamma_s^+$ of a free relativistic particle of mass $m>0$ and spin $s>0$ as a 9-dimensional Poincaré and space reflection invariant submanifold of the 12-dimensional surface $\mathcal{J}_{ms}^+ \subset \mathcal{J}_{14}$ given by the Casimir constraints

$$W^2 + p^2 s^2 = 0 \ . \tag{3.2a}$$

We have

$$m^2 + p^2 = 0 \qquad , \qquad p^0 > 0 \tag{3.2b}$$

W being the Pauli-Lubanski vector

$$W_\varkappa = \tfrac{1}{2} \cdot \varepsilon_{\varkappa\lambda\mu\nu} J^{\lambda\mu} p^\nu \quad ; \quad (\varepsilon^{0123} = \varepsilon_{1230} = 1) \tag{3.3a}$$

$$W^2 = V^2 - \tfrac{1}{2} p^2 S_{\mu\nu} S^{\mu\nu} \tag{3.3b}$$

where

$$S_{\mu\nu} = J_{\mu\nu} - L_{\mu\nu} \quad ; \qquad L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \tag{3.4a}$$

$$V_\mu = S_{\mu\nu} p^\nu \ . \tag{3.4b}$$

The three additional relations will be derived from the following requirement.

Introducing the Hamiltonian constraint

$$H = \frac{\lambda_1}{2} (m^2 + p^2) + \frac{\lambda_2}{2} (W^2 + p^2 s^2) \approx 0 \tag{3.5}$$

where λ_1 and λ_2 are arbitrary Lagrange multipliers, we demand that the particle world line on $\mathcal{M}_{ms}$ is independent of λ_1 and λ_2. Since for $\lambda_2 = 0$ we have $\dot{x} = \{x, H\} = \lambda_1 p$, it follows that the vectors $\dot{x}$ and p should be colinear (i.e. $[\dot{x}] = [p]$) for any λ_1, λ_2. The necessary and sufficient condition for such a "gauge invariance" of world lines is

$$\{x, W^2\} \approx C_p \qquad (\text{i.e. } \{x, W^2\}\big|_{\mathcal{M}_{ms}} = C_p) \ . \tag{3.6}$$

It turns out that this requirement, together with the condition of space reflection invariance determines the mass-spin shell completely.

Let ${}^*S_{\varkappa\lambda}$ be the dual tensor to S :

$${}^*S_{\varkappa\lambda} = \frac{1}{2} \varepsilon_{\varkappa\lambda\mu\nu} S^{\mu\nu} . \tag{3.7a}$$

Then the $\mathfrak{sl}(2, \mathbb{C})$ Casimir operators

$$S_{\mu\nu} S^{\mu\nu} = - {}^*S_{\mu\nu} {}^*S^{\mu\nu} \qquad {}^*S^{\mu\nu} S_{\mu\nu} \tag{3.7b}$$

have zero Poisson brackets with all dynamical variables (i.e. with all

functions on $\mathcal{T}_{14}$).

Comment. The existence of a non-trivial centre of the Lie algebra of Poisson brackets on $\mathcal{T}_{14}$ (consisting of all/smooth/functions of the variables (3.7b)) indicates that $\mathcal{T}_{14}$ is not a symplectic manifold. More precisely, there is no symplectic form on $\mathcal{T}_{14}$ which corresponds to the Poisson brackets (3.1).

According to (3.3), (3.4), (3.1) Eq. (3.6) implies

$$V_\mu S^{\mu\nu} \approx C_1 p^\nu . \tag{3.8}$$

In order to exploit Eq. (3.8) we shall establish the identity

$$*S^{\lambda\mu} S_{\lambda\nu} = S^{\lambda\mu} *S_{\lambda\nu} = \tfrac{1}{4} *S^{\varkappa\lambda} S_{\varkappa\lambda} \delta^\mu_\nu . \tag{3.9}$$

To this end we shall use the fact that for every bivector there exists an (orthonormal) basis (e^0, e^1, e^2, e^3) in Minkowski space such that[+)]

$$S_{\mu\nu} = f_{01}(e^0_\mu e^1_\nu - e^0_\nu e^1_\mu) + f_{23}(e^2_\mu e^3_\nu - e^3_\mu e^2_\nu) \tag{3.10a}$$

and hence,

$$*S_{\mu\nu} = f_{01}(e^2_\mu e^3_\nu - e^2_\nu e^3_\mu) - f_{23}(e^0_\mu e^1_\nu - e^0_\nu e^1_\mu) \tag{3.10b}$$

$$*S^{\lambda\mu} S_{\lambda\nu} = f_{01} f_{23}(e^1{}^\mu e^1_\nu + e^2{}^\mu e^2_\nu + e^3{}^\mu e^3_\nu - e^0{}^\mu e^0_\nu) = f_{01} f_{23} \delta^\mu_\nu . \tag{3.11}$$

We deduce Eq. (3.9). Multiplying both sides of (3.8) by $*S_{\nu\lambda}$ and using (3.9), we find

$$\tfrac{1}{4} *S^{\lambda\mu} S_{\lambda\mu} V_\nu \approx C_1 W_\nu$$

Introducing the pseudoscalar ψ by

$$\tfrac{1}{4} *S^{\varkappa\lambda} S_{\varkappa\lambda} = \psi s^2 \tag{3.12a}$$

and using the equation $V^2 \approx C_1 p^2$ and the relation (3.3) we deduce that

$$C_1 = -\psi^2 s^2 , \qquad \tfrac{1}{2} S_{\mu\nu} S^{\mu\nu} \approx (1-\psi^2) s^2 . \tag{3.12b}$$

[+)] See, e.g., T6) Vol. II Sec. VII.1. corollary 1.5 (p.351).

Thus we end up with the constraint

$$V_\nu + \psi W_\nu \approx 0 \,. \tag{3.13}$$

Finally, invariance under space reflections implies

$$\psi = 0 \approx {}^*S^{\varkappa\lambda} S_{\varkappa\lambda} \tag{3.14}$$

and hence

$$\tfrac{1}{2} S^{\mu\nu} S_{\mu\nu} \approx s^2 \tag{3.15}$$

$$V_\mu = S_{\mu\nu} p^\nu \approx 0 \,. \tag{3.16}$$

These are the additional constraints which complete the definition of $\mathcal{M}_{ms}$. (Note that only three of the four constraints (3.16) are independent, because of the strong equation pV=0). The existence of a non-trivial solution p of the system of equations (3.16) implies the vanishing of the determinant of the matrix $(S_{\mu\nu})$

$$\det(S_{\mu\nu}) = (S_{01}S_{23} + S_{02}S_{31} + S_{03}S_{12})^2 = \left(\tfrac{1}{4} {}^*S_{\mu\nu} S^{\mu\nu}\right)^2 \approx 0 \,. \tag{3.17}$$

Hence Eq. (3.14) is a consequence of the constraints (3.16) (a fact which also follows from (3.9)).
For $p^2 < 0$ the rank of the matrix of Poisson brackets

$$\{V_\mu, V_\nu\} = p^2 S_{\mu\nu} + p_\mu V_\nu - p_\nu V_\mu \approx p^2 S_{\mu\nu} \tag{3.18}$$

is 2(the constraint (3.14) being first class). Eq. (3.14) (or (3.17)) implies that $S_{\mu\nu}$ is a decomposable bivector. The constraints (3.16) can be solved by writing

$$S_{\varkappa\lambda} = e_{\varkappa\lambda\mu\nu} \frac{W^\mu p^\nu}{-p^2} \tag{3.19a}$$

or, equivalently,

$$p^2 {}^*S_{\mu\nu} = W_\mu p_\nu - W_\nu p_\mu \,. \tag{3.19b}$$

For $S_{\mu\nu}$ given by (3.19a) the product

$$\mathfrak{S}^\mu{}_\nu = \frac{1}{s^2} S^{\lambda\mu} S_{\lambda\nu} \tag{3.20a}$$

is a 2-dimensional projection operator:

$$\mathfrak{S}^\mu{}_\nu = \delta^\mu{}_\nu - \frac{p^\mu p_\nu}{p^2} - \frac{W^\mu W_\nu}{W^2} \tag{3.20b}$$

(so that $\mathfrak{S}p = 0 = \mathfrak{S}W$) . It is straightforward to verify that

$$\frac{1}{s^2}\left(S^{\lambda\mu} S_{\lambda\nu} - {}^*S^{\lambda\mu} {}^*S_{\lambda\nu}\right) = \delta^\mu{}_\nu \,.$$

We are now ready to define the 10 dimensional symplectic manifold Γ_s^+ which plays the role of phase space of a "conformal particle" of spin s and positive energy, but unrestricted (positive) mass:

$$\Gamma_s^+ = \{(x,p,S_{\mu\nu}) \in \mathcal{T}_{14} \; ; \; V_\mu = 0 \, , \; \tfrac{1}{2} S_{\mu\nu} S^{\mu\nu} = s^2 \, , \; p^0 > |\underline{p}| \} = \tag{3.21a}$$

$$= \{(x,p,W) \in \mathbb{R}^{12} ; \; pW = 0 \, , \; W^2 + s^2 p^2 = 0 \, , \; p^0 > |\underline{p}| \} \, . \tag{3.21b}$$

According to (3.18) the definition (3.21a) of Γ_s^+ involves a pair of second class constraints. The bracket structure on Γ_s^+ is obtained by a modification of the original Poisson brackets due to Dirac.

Let Γ be the 12-dimensional submanifold of $\mathcal{T}_{14}$ obtained by imposing the first class constraints (3.14) and (3.15) and let $\Gamma_* = \Gamma_s^+$. The Dirac brackets $\{f,g\}_*$ are characterized by the following requirements (see D3) H1)).

(a) They have the algebraic properties of a commutator and satisfy the Leibniz rule for differentiation:

$$\{f,gh\}_* = g\{f,h\}_* + h\{f,g\}_* \, .$$

(b) The constraints (in our case (3.16)) have zero Dirac brackets with any dynamical variable:

$$\{f, V_\mu\}_* = 0 \, . \tag{3.22}$$

(c) If f is an invariant observable on Γ_s^+ (i.e. if it has weakly vanishing Poisson brackets with the constraints), then its Dirac brackets with any dynamical variable coincide with the corresponding Poisson bracket in Γ :

$$\{f, V_\mu\} \approx 0 \quad \Rightarrow \quad \{f,g\}_* \approx \{f,g\} \, . \tag{3.23}$$

These conditions determine the Dirac brackets uniquely. In order to construct $\{f,g\}_*$ in the case at hand, we observe that according to (3.18) and (3.20)

$$-\frac{S^{\lambda\nu}}{p^2 s^2} \{V_\lambda, V_\nu\} \approx \delta^\kappa{}_\nu \, .$$

It is now a simple exercise to verify that the brackets

$$\{f,g\}_* = \{f,g\} + \{f,V^\kappa\} \frac{S_{\kappa\nu}}{p^2 s^2} \{V^\nu, g\} \tag{3.24}$$

satisfy all requirements (a-c) of the definition of Dirac brackets.

Since the Poincaré generators are invariant observables (in the sense of condition (c) their Dirac brackets with any dynamical variable coincide with their Poisson brackets (see (3.1)). For the Dirac brackets of x^μ and $S^{\mu\nu}$, on the other hand, we find

$$\{x^\mu, x^\nu\}_* = \frac{S^{\mu\nu}}{-p^2} \tag{3.25}$$

$$\{x^\lambda, S^{\mu\nu}\}_* \approx \frac{1}{-p^2}\left(p^\mu S^{\lambda\nu} - p^\nu S^{\lambda\mu}\right) \tag{3.26}$$

$$\{S_{\varkappa\lambda}, S_{\mu\nu}\}_* = \Pi_{\varkappa\mu} S_{\lambda\nu} - \Pi_{\varkappa\nu} S_{\lambda\mu} + \Pi_{\lambda\nu} S_{\varkappa\mu} - \Pi_{\lambda\mu} S_{\varkappa\nu} \tag{3.27}$$

$$\Pi_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \ . \tag{3.28}$$

Summarizing the results of this section we would like to stress the following point. The assumption about gauge invariance of world lines (expressed by Eq. (3.6)) has led us to the constraints (3.16) which imply that the position observables of a spinning particle cannot be canonical (their brackets being instead given by (3.25)). This property of physical position variables has first been pointed out by Pryce in 1948 (see P5)).

For other work on relativistic spinning particles see, e.g. W1).

4. Generalized N-particle mass shell

4A. Definition. The fibre bundle $\mathcal{M} \to \Gamma_*$

The large phase space Γ^N of a system of N (spinless) particles is taken as the direct product of single particle phase spaces of Sec. 2: $\Gamma^N = \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_N$.

It is equipped with the (Poincaré invariant) symplectic form

$$\omega = \sum_{a=1}^{N} \omega_a = \sum_{a=1}^{N} dq_a^\mu \wedge dp_{a\mu} \ . \tag{4.1}$$

The generalized N-particle mass shell is defined as a 7 N-dimensional connected Poincaré invariant submanifold $\mathcal{M}$ of Γ^N with the properties listed below.

(i) In any Lorentz frame the surface $\mathcal{M}$ is locally given by N canonical equations of the form

$$\varphi_a^c \equiv h_a(\underline{p}_1, \ldots, \underline{p}_N; q_{12}, \ldots, q_{N-1\,N}) - p_a^\circ = 0 \ ; \ a = 1, \ldots, N. \tag{4.2}$$

Here the relative co-ordinates $q_{ab} = q_a - q_b$ are required to be space-

like

$$q_{ab}^2 > 0 \qquad \text{for} \quad 1 \le a < b \le N \tag{4.3}$$

while the total momentum

$$P = \sum_{a=1}^{N} p_a \tag{4.4}$$

is assumed to be positive time-like:

$$-P^2 = P_0^2 - \underline{P}^2 = W^2 > 0, \qquad P^0 \left(\approx \sum_a h_a\right) > 0. \tag{4.5}$$

(ii) The equations defining $\mathcal{M}$ are <u>compatible</u> with respect to the Poisson bracket structure on Γ^N:

$$\{\varphi_a^c, \varphi_b^c\} = \frac{\partial h_b}{\partial q_a^o} - \frac{\partial h_a}{\partial q_b^o} + \{h_a, h_b\} = 0 \,. \tag{4.6}$$

(In the terminology of Dirac, φ_a^c satisfying this property are called <u>first class constraints</u>.) To be precise, we shall adopt the following stronger requirement (whose formulation makes use of a more sophisticated mathematical language).

Let $\mathrm{Ker}\,\omega|_{\mathcal{M}}$ be the set of all vectors tangent to $\mathcal{M}$, on which the restriction $\omega|_{\mathcal{M}}$ of the symplectic form (4.1) vanishes. (If $\mathcal{M}$ is given locally by the set of equations $\varphi_a = 0$, $a = 1, \ldots, N$, then $\mathrm{Ker}\,\omega|_{\mathcal{M}}$ is generated by the Liouville operators

$$L_{\varphi_a} = \sum_{b=1}^{N} \left(\frac{\partial \varphi_a}{\partial p_b} \cdot \frac{\partial}{\partial q_b} - \frac{\partial \varphi_a}{\partial q_b} \cdot \frac{\partial}{\partial p_b} \right), \quad a = 1, \ldots, N \tag{4.7}$$

also called Hamiltonian vector fields.) We assume that it is an N-dimensional integrable vector sub-bundle[+)] of the tangent bundle $T\mathcal{M}$, and the foliation

$$\mathcal{M} \longrightarrow \Gamma_* = \mathcal{M} / \mathrm{Ker}\,\omega|_{\mathcal{M}} \tag{4.8}$$

is a (locally trivial) fibre bundle. The 6 N-dimensional base space of this fibre bundle is the (gauge invariant!) physical <u>phase space</u>.

+) A vector sub-bundle $\mathcal{V}$ of a tangent bundle is <u>integrable</u> if the commutator $[X,Y]$ of any two sections of $\mathcal{V}$ is again a section of $\mathcal{V}$. If $\mathcal{M}$ is given by the first class constraints $\varphi_a(\gamma) = 0$, then the points γ_* of Γ_* can be identified with the N-dimensional integral surfaces $\gamma = \gamma(\sigma_1, \ldots \sigma_N)$ of the system of partial differential equations $\frac{\partial \gamma}{\partial \sigma_a} = L_{\varphi_a} \gamma$ on $\mathcal{M}$.

(iii) In order to have a standard scattering theory we require a separability (or cluster decomposition) property which says, in a physical language, that clusters of particles separated by large space-like intervals do not interact. We shall not attempt to give the most general geometric formulation of this assumption but will just note that it implies, in particular,

$$\lim_{q_a \to \infty} h_a = \sqrt{m_a + \underline{p}_a^2} \tag{4.9}$$

whenever the constraints (4.2) are defined globally (m_a being the mass of particle a) .

Comments and remarks

1) It should be stressed that we define a relativistic Hamiltonian system by the surface $\mathcal{M}$ (and the form ω) and not by the specific (local) equations which describe $\mathcal{M}$. Indeed, the same surface can be given in terms of different sets of constraints and that should not affect the physics. We shall actually exploit the freedom in writing down the equations for $\mathcal{M}$ in various ways, depending on the problem we are dealing with. For the general discussion of this section and for comparison with the Curie-Jordan-Sudarshan (CJS) approach C4) (see Sec. 7B below), it is convenient to use the set (4.2) of equations solved with respect to the particles' energies. It has the drawback, however, of not being manifestly Lorentz invariant. (It can be made manifestly Euclidean /and time-translation/ invariant by assuming that the functions φ_a^c only depend on the scalar products of their 3-dimensional vector arguments.) In order to ensure the assumed Lorentz invariance of $\mathcal{M}$ we have to demand the (weak) vanishing of the Poisson brackets of φ_a^c with the Lorentz boosts $J_{oi} = -J^{oi}$ where

$$J^{\mu\nu} = \sum_{a=1}^{N} (q_a^\mu p_a^\nu - q_a^\nu p_a^\mu) . \tag{4.10}$$

This leads to a set of (strong) non-linear partial differential equations for the functions h_a :

$$\sum_{a=1}^{N} \left(\underline{q}_a \frac{\partial}{\partial q_a^0} + q_a^0 \frac{\partial}{\partial \underline{q}_a} + h_a \frac{\partial}{\partial \underline{p}_a} \right) h_b - \underline{p}_b = 0 \tag{4.11}$$

(In the canonical gauge, in which all q_a^0 are equal /see Eq. (4.14) below/, the second term under the summation sign vanishes). In describing the set of admissible 2-particle constraints (Sec. 5A) we shall use instead a manifestly covariant form of φ_a .

2) In the 2-particle case we add the condition

$$P p_a < 0 \qquad , a = 1, 2$$

(meaning that particle energies are positive in the centre of mass frame) to the inequalities (4.5) that determine the range of particle momenta on $\mathcal{M}$.

3) The strong equation (4.6) is a consequence of the weak equality $\{\varphi_a^c, \varphi_b^c\} \approx 0 \; (= \{\varphi_a^c, \varphi_b^c\}|_{\mathcal{M}})$ for φ_a^c given by the canonical expression (4.2) . Our second requirement ensures the reparametrization invariance of the theory (which involves, a priori, an N-dimensional manifold of evolution parameters).

4) The qualification "local" concerning the canonical equations of the constraints means that we would not like to exclude <u>a priori</u> the possibility that $p_a^{\,o}$ are multivalued functions of the remaining variables.

5) Theories in which only conditions (i) and (ii) (but not necessarily (iii)) are required and which give room to confining potentials (e.g., of the type studied in L4) can, of course, also be considered. As already stated, the separability requirement (iii) is only needed to ensure the existence of scattering states and of an S-matrix. In particular, it will not be used in our discussion of gauge dependence of particle world lines (see Sec. 7).

6) If we identify physics with particle world lines and associated asymptotic representations of the Poincaré group (but regard the choice of relative momenta in the interaction region as a matter of convention), then different submanifolds $\mathcal{M}$ of Γ may correspond to physically equivalent dynamics. Two generalized mass shells $\mathcal{M}$ and $\bar{\mathcal{M}}$ are considered as <u>physically equivalent</u> if for any fixed choice of the Hamiltonian $H = \sum_a \lambda_a \varphi_a$ they lead to the same world lines for the same initial conditions, and if, in addition, they give rise to the same realization of the Poincaré group. If the mass shell $\mathcal{M}$ is separable (that is, if requirement (iii) takes place) then we demand that $\bar{\mathcal{M}}$ is also separable and that the difference of the corresponding canonical constraints $\varphi_a^c - \bar{\varphi}_a^c$ vanishes for $\underline{q}_a \longrightarrow \infty$. A family of physically equivalent generalized mass shells is obtained by applying to a given $\mathcal{M}$ all canonical transformations of the type

$$q_a \longrightarrow \bar{q}_a = q_a \quad , \quad p_a \longrightarrow \bar{p}_a = p_a + \frac{\partial}{\partial q_a} F\left(\tfrac{1}{2} q_{12}^2, \ldots, \tfrac{1}{2} q_{N-1\,N}^2\right). \qquad (4.12)$$

Because of the Poincaré invariance of F the generators of the Poincaré group do not change under a canonical transformation:

$$P = \sum_a p_a = \sum_a \bar{p}_a \; ; \qquad J^{\mu\nu} = \sum_a (q_a^\mu p_a^\nu - q_a^\nu p_a^\mu) = \sum_a (\bar{q}_a^\mu \bar{p}_a^\nu - \bar{q}_a^\nu \bar{p}_a^\mu) .$$

It can be proved that locally the transformations of type (4.12) are the most general ones with all these properties. If, in addition, $F \to 0$ and $\partial F / \partial q_a \to 0$ for $q_a \to \infty$ then the asymptotic momenta also coincide: $\bar{p}_a^{as} = p_a^{as}$. In the Lagrangian picture a transformation of the form (4.12) corresponds to the addition of a total time derivative to the Lagrangian.

7) A generalization of the above scheme is also possible, which includes manifolds $\mathcal{M}$ with a boundary. An example of this type is the elastic scattering of rigid balls (see M3)). Singular potentials with singularities on lower dimensional manifolds (like the Coulomb potential, considered in Sec. 5B) are included in our scheme. The singularities of the functions h_a (or φ_a) have to be excluded from $\mathcal{M}$ (which is not assumed to be a closed manifold). In particular, Eq. (4.3) excludes coinciding points[+] from $\mathcal{M}$ ($q_a \neq q_b$ for $a \neq b$). Unphysical (say, strong attractive) singularities are automatically discarded by the integrability condition involved in assumption (ii).

4B. Gauges and Hamiltonians

A Hamiltonian H of an N-particle system is defined as a linear combination of the constraints φ_a^c with positive (γ-dependent) coefficients ("Lagrange multipliers"). The Lagrange multipliers can be determined up to an overall factor by giving N-1 gauge conditions (whose physical role is to pick up an equal time surface in $\mathcal{M}$) and demanding that they have (weakly) vanishing Poisson brackets with the Hamiltonian.

It will be sufficient for our purposes to consider gauge conditions of the type

$$n q_{ab} = 0 \quad , \qquad 1 \leq a < b \leq N \tag{4.13}$$

where n is a time-like vector; for $N=2$ we shall assume n to have zero Poisson brackets with $q = q_{12}$ (and thus it will only be allowed to depend on the total momentum P in that case). If n is a constant vector then we can choose the Lorentz frame in such a way that the time axis points along n , so that conditions (4.13) assume the form

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[+] Such an assumption is justified because the known physical interactions (like electromagnetic and gravitational) are singular for coinciding arguments. One can, however, consider a more general scheme as well, in which the inequalities (4.3) do not take place (see M3).

$$q^0_{ab} = 0 \quad , \qquad a,b = 1,\ldots,N . \tag{4.14}$$

The most general Hamiltonian constraint is then proportional to the sum H^C of canonical constraints:

$$H^C = \sum_{a=1}^{N} \varphi_a{}^C = h - P^0 \ (\approx 0) \quad \text{where} \quad h = \sum_{a=1}^{N} h_a \ . \tag{4.15}$$

The remaining freedom is fixed by the choice of the time scale. The Hamiltonian H^C corresponds, in particular, to an evolution parameter t equal (up to a common additive constant) to the zeroth component of each of the position 4-vectors:

$$t - t^0 = q^0_1 = \ldots = q^0_N \tag{4.16}$$

Indeed, by definition, t is an evolution parameter, corresponding to a Hamiltonian H, if for any dynamical variable f we have

$$\frac{df}{dt} = \{f, H\} .$$

For $H = H^C$ and $f = q^0_a$ we find (according to (4.2) and (4.15)

$$\frac{dq^0_a}{dt} = \{q^0_a, H^C\} = \{q^0_a, -P^0\} = -\eta^{00} = 1 \tag{4.17}$$

so that $q^0_a = t + C_a$. Finally, the constants of the motion C_a have to be equal because of (4.14). (Note that a more general gauge condition of the type $q^0_{ab} = C_{ab} \neq 0$ is excluded by the requirement that the vectors q_{ab} are space-like on $\mathcal{M}$.)

Thus the gauge is fixed by giving the equal time surface (of type (4.13)), which specifies the relative gauge, and the time scale (an example of which is provided, say, by the first equation (4.16)). We shall say that a dynamical variable f is an observable, if its time evolution is independent of the choice of relative gauge, i.e., if

$$\{f, H^{(n)}\} = \lambda(n) \{f, H^C\} \quad \text{for} \quad \{n q_{ab}, H^{(n)}\} = 0 \ . \tag{4.18}$$

It is easily proved that f is an observable, if its Poisson brackets with the difference

$$\varphi_a{}^C - \varphi_{a+1}{}^C \ (\approx 0) \quad , \qquad a = 1,\ldots,N \tag{4.19}$$

vanish (weakly).

A gauge invariant observable is an observable which is at the same time a constant of the motion. The generators of the Poincaré group

P_μ and $J_{\mu\nu}$ are always gauge invariant observables (because of the assumed Poincaré invariance of $\mathcal{M}$).

In geometric language a gauge specifies for each value of the evolution parameter a section of the fibre bundle $\mathcal{M} \to \Gamma_*$ (4.8).

5. Admissible two-particle interactions

5A. Manifestly invariant compatible constraints for N=2

In order to exhibit a class of generalized 2-particle mass shells it is useful to take Lorentz invariance explicitely into account and to look for a pair of manifestly Poincaré invariant constraints

$$\varphi_a = \frac{1}{2}(m_a^2 + p_a^2) + \phi_a \approx 0 \quad , \qquad a = 1, 2 \tag{5.1}$$

satisfying the compatibility condition

$$\{\varphi_1, \varphi_2\} \approx 0 \, . \tag{5.2}$$

Let us first work out some 2-particle kinematics.

We define the relative momentum

$$p = \mu_2 p_1 - \mu_1 p_2 \qquad (\mu_1 + \mu_2 = 1) \tag{5.3}$$

requiring that

$$\varphi \equiv Pp = \frac{1}{2}\left(m_1^2 + p_1^2 - m_2^2 - p_2^2\right) . \tag{5.4}$$

Taking P and p as independent variables, we can write p_1 and p_2 in the form

$$p_1 = \mu_1 P + p \quad , \quad p_2 = \mu_2 P - p \, . \tag{5.5}$$

Inserting (5.5) into (5.4) we find that

$$\mu_1 - \mu_2 = \frac{m_1^2 - m_2^2}{W^2} \quad , \quad \text{so that} \quad \mu_{1,2} = \frac{1}{2} \pm \frac{m_1^2 - m_2^2}{2W^2} \, . \tag{5.6}$$

The compatibility condition (5.2) becomes more tractable in terms of the relative co-ordinates

$$q = q_1 - q_2 \tag{5.7}$$

and p , and the following linear combinations of the original constraints:

$$\varphi_1 - \varphi_2 = \varphi + D \ (\approx 0) \quad , \quad D = \phi_1 - \phi_2 \qquad (\varphi = Pp) \tag{5.8}$$

$$H = \mu_2\varphi_1 + \mu_1\varphi_2 = \frac{1}{2}\left(p^2 - b^2(w)\right) + \phi\,(\alpha 0)\ ;\quad \phi = \mu_2\phi_1 + \mu_1\phi_2 \qquad (5.9)$$

where

$$b^2(w) = \frac{1}{4w^2}\left[w^4 - 2\,(m_1^2+m_2^2)w^2 + (m_1^2-m_2^2)^2\right] \qquad (5.10)$$

is the value of the relative momentum square on the free 2-particle mass shell. Inserting $\varphi_1 = H + \mu_1(\varphi+D)$ and $\varphi_2 = H - \mu_2(\varphi+D)$ in Eq. (5.2), we find

$$\{\varphi_1,\varphi_2\} = \{\varphi+D, H\} = -P\cdot\frac{\partial\phi}{\partial x} + p\,\frac{\partial D}{\partial x} + \{D,\phi\} \approx 0\,. \qquad (5.11)$$

For given D the corresponding strong equation can be used to determine the general admissible form of the interaction function ϕ. (It is a first order linear partial differential equation for ϕ, whose solution involves a functional freedom.) For most of these lectures we shall consider the simple special case in which

$$D = 0 = P\cdot\frac{\partial\phi}{\partial q}\ . \qquad (5.12)$$

The last equation is satisfied by any function ϕ which depends on q through

$$q_\perp = q + (q\hat{P})\hat{P} \quad \text{where} \quad \hat{P} = \frac{1}{w}P \quad (\hat{P}^2 = -1)\ . \qquad (5.13)$$

The solution thus obtained is general enough to accomodate (in its quantized version, including spin) the quasipotential equations considered so far (see, for a review, R4), and Sec. 8 of these notes). A special solution with $D\neq 0$ is given by Eq. (7.1) below.

Remarks

1) The functions ϕ_a in the constraints (5.1) are not fixed uniquelly by the surface $\mathcal{M}$ even if we assume that they go to zero for large $|q_\perp|$. This is clear from the example of the constraints

$$\varphi_1 = \frac{1}{2}(m_1^2+p_1^2) + \rho_1(q_\perp^2)(m_2^2+p_2^2)\ ,\quad \varphi_2 = \frac{1}{2}(m_2^2+p_2^2) + \rho_2(q_\perp^2)(m_1^2+p_1^2)$$

where $\rho_1 \neq \rho_2$ (and $\lim_{q_\perp\to\infty}\rho_a = 0$) which describe the free particle mass shell in disguise. In order to make the separation of the interaction term in (5.1) unique we shall assume that ϕ_a may only depend on the relative momentum p (4.22) through the angular momentum

$$\ell^2 = \frac{1}{2}\left(q_\perp^\mu p^\nu - q_\perp^\nu p^\mu\right)\left(q_{\perp\mu}p_\nu - q_{\perp\nu}p_\mu\right) = q_\perp^2 p^2 - (q_\perp p)^2 \qquad (5.14)$$

apart from a possible linear dependence in pq. Under this assumption the relevant (Poincaré invariant) solution of (4.31) can be written in the form

$$\phi = (\phi_1 = \phi_2 =) = A(r, w, \ell) + pq\, B(r, w, \ell) \tag{5.15}$$

where

$$r = |q_{\perp}| = \sqrt{q^2 + (q\hat{P})^2}\,. \tag{5.16}$$

Note that in the examples of Secs. 5B and 8C we deal with a still simpler interaction term

$$\phi = \phi(r, w)\,. \tag{5.17}$$

2) The requirement (i) of Sec. 4A, which asserts the existence of the canonical form (4.2) of the constraints, sets an additional restriction on ϕ_a :

$$d = \det\left(\frac{\partial \varphi_a}{\partial p_b^0}\right) > 0 \qquad \text{for} \qquad P^0 > |\underline{P}|\,. \tag{5.18}$$

(The sign of the determinant is chosen to fit the free-particle case and hence the limit $r \to \infty$; it cannot change with r since d should not vanish anywhere.) In the special case when $\phi_1 = \phi_2 = \phi$ has the form (5.17) Eq. (5.18) gives

$$p_1^0 p_2^0 - P^0 \frac{\partial \phi}{\partial P^0} > 0 \tag{5.19}$$

where

$$\frac{\partial \phi}{\partial P^0} = \hat{P}^0 \frac{\partial \phi}{\partial w} - \frac{P q}{w^2 r} \cdot q_{\perp}^0 \frac{\partial \phi}{\partial r}\,.$$

Going to the centre of mass frame we find, in particular (for large r),

$$\mu_1 \mu_2 > 0 \quad \text{or} \quad w^2 > |m_1^2 - m_2^2| \qquad (\text{i.e. } Pp_K < 0)\,. \tag{5.20}$$

These inequalities are, however, not automatic for the bounded motion (for which one cannot go to the large r limit).

5B. Relativistic reduced mass and minimal Coulomb interaction

One qualitative reason why a 2-particle system is so much simpler to study (even in the non-relativistic case) than a system of N interacting particles for $N \geq 3$, is displayed by the possibility of reducing the study of two spinless particles to an effective one-parti-

cle problem in an external field. (There does not seem to be any comparable reduction for the 3-particle problem.)

The key notion in studying the relative motion of a non-relativistic 2-particle system is the concept of a reduced mass m, satisfying the equation

$$m M = m_1 m_2 \tag{5.21}$$

where M is the total mass of the 2-particle system ($M = m_1 + m_2$ for non-relativistic particles). We shall extend this concept to the relativistic case by simply replacing M with the relativistic total mass $w = \sqrt{P^2}$. Thus we come to the notion of a relativistic reduced mass, m_w, given by

$$m_w = \frac{m_1 m_2}{w} . \tag{5.22}$$

We shall identify the remaining characteristics of the effective particle and of the "external field" in the centre of mass frame (the effective "external source" being at rest just in that frame). We shall work with the special 2-particle constraints (5.12) (5.17) throughout this section so that we shall have, in particular,

$$\varphi = pP \approx 0 . \tag{5.23}$$

Note first of all that the relative momentum p and the co-ordinate $q_\perp$ have zero time components in the rest frame of P :

$$p = (0, \underline{p}) \quad , \quad q_\perp = (0, \underline{r}) \quad \text{for} \quad P = (w, \underline{0}) . \tag{5.24}$$

Recalling that the on-shell value of $\underline{p}^2$ is $b^2(w)$ (5.10) we define the energy E of the effective particle of mass m_w and 3-momentum $\underline{p}$ by

$$E = \sqrt{m_w^2 + \underline{p}^2} = \frac{1}{2w}\left(w^2 - m_1^2 - m_2^2\right) > 0 . \tag{5.25}$$

(Note that the positivity of the right-hand side of (5.25), which we assume, is a stronger restriction on the value of w than the requirement (5.20) of the preceding subsection).

It is easily verified that the free effective particle constraint

$$H_0 \equiv \frac{1}{2}\left(m_w^2 + \underline{p}^2 - E^2\right) \left[= \frac{1}{2}\left(p^2 - b^2(w)\right)\right] \approx 0 \tag{5.26}$$

preserves the Markov-Yukawa gauge condition

$$P_q = 0 \quad . \tag{5.27}$$

Consider a system of two charged particles (of charges e_1 and e_2). In the nonrelativistic limit their interaction is described by the Coulomb potential

$$V(r) = \frac{e_1 e_2}{4\pi r} \quad . \tag{5.28}$$

A naive way to combine (6.6) and (6.8) is to try a minimal type "relativistic Coulomb interaction" by writing down the Hamiltonian constraint

$$H_{Coul} = \frac{1}{2}\left[m_w^2 + \underline{p}^2 - (E-V)^2\right] \approx 0 \quad . \tag{5.29}$$

Surprising as it may look, it turns out that this expression leads indeed to correct results provided that r is not too small. Actually, as it will be shown in Sec. 8C in the framework of the quasipotential approach to quantum electrodynamics, the electromagnetic interaction "Hamiltonian" H_{em} , derived from the 1-photon exchange diagram, differs from H_{Coul} only by a term of the order of $(e_1 e_2)^2\ (wr^2)^{-2}$ for $r > o$ (it comes from the square of the vector potential):

$$H_{em} = H_{coul} + \frac{1}{8w^2r^2}V^2 = \frac{1}{2}(\underline{p}^2 - b^2) + EV + \frac{1}{2}V^2\left(\frac{1}{4w^2r^2} - 1\right) \approx 0 \quad . \tag{5.30}$$

Although the addition to H_{coul} is negligible classically at large distances and leads to higher order corrections to the quantum energy levels if treated perturbatively, it affects in a qualitative way the short distance behaviour of the interaction function

$$\phi_{em} = EV + \frac{1}{2}V^2\left(\frac{1}{4w^2r^2} - 1\right) = \phi_{coul} + \frac{\alpha^2}{8w^2r^4} \tag{5.31}$$

$$\phi_{coul} = H_{coul} - H_0 = EV - \frac{1}{2}V^2$$

where α is the fine structure constant:

$$\alpha = \frac{|e_1 e_2|}{4\pi} \quad . \tag{5.32}$$

Indeed, the leading singularity of ϕ_{Coul} for small r , $-\frac{\alpha^2}{2r^2}$ is attractive and presents a problem for small angular momentum ($l^2 < \alpha^2$). The correction $\phi_{em} - \phi_{coul}\left(= \frac{\alpha^2}{8w^2r^4}\right)$, being dominant at small distances and repulsive, makes the entire interaction term bounded below. It may also lead to new (non-perturbative) physical results in problems in which short distances become relevant (cf. B2)).

For fixed w the constraint

$$H = \frac{1}{2}(\underline{p}^2 - b^2(w)) + \phi(r,w) \qquad (5.33)$$

has the same structure (in particular, the same p-dependence) as a non-relativistic Hamiltonian. This observation allows to extend known non-relativistic results, including exact solutions, to the relativistic case.

6. Relativistic addition of interactions

6A. The problem

For a non-relativistic N-particle system an interaction that only involves two body forces is defined as a simple sum of 2-particle potentials. For a relativistic system such a simple minded procedure runs into conflict with compatibility. For instance 3-particle constraints of the form

$$\varphi_1 = \frac{1}{2}(m_1^2 + p_1^2) + \phi_{12} + \phi_{13} \approx 0 \qquad \text{and cyclic permutations}$$

where

$$\phi_{ab} = \phi(r_{ab}, w_{ab}) \quad , \quad r_{ab}^2 = q_{ab}^2 + (\hat{P}_{ab} q_{ab})^2 \quad , \quad P_{ab} = p_a + p_b = w_{ab} \hat{P}_{ab} \qquad (6.1)$$

(as proposed in C2)) are not first class. One has to add appropriate many particle interactions to the sum of 2-body forces in order to obtain a consistent relativistic theory. We shall present two types of solutions to this problem.

First, we shall work out (in Sec. 6B) an extension to the (classical) generalized mass shell framework of Sokolov's (quantum mechanical) procedure for relativistic addition of interactions (see S4), S6)). As the argument is not very constructive (since it assumes the knowledge of Moeller's 2-particle wave operators) we also present (in Sec. 6C) an iterative solution (similar but not identical to the one given by Sazdjian S3)).

Remark. The separability condition for a 3-particle system in the presence of 2-body forces implies that when one of the particles is taken to infinity the remaining two continue to interact. If this (strong) separability condition is abandoned, then one can construct compatible 3-particle constraints of a rather uninteresting type (see Sec. 6A of ref. T5)). Mutze M4) has argued (in an alternative formulation of the relativistic many body problem) that only N-particle separable forces can appear among N interacting particles (if one of the particles when taken to infinity becomes free then the remaining N-1 particles also

become free). It appears that the assumptions underlying this result are unnecessarily strong.

6B. Separable N-particle interactions in terms of classical wave operators

Let H and H_0 be two Hamiltonians (i.e. two (Euclidean invariant) functions on phase space) and let L_H and L_{H_0} be the corresponding Liouville operators (4.7). The classical wave (or Moeller) operators are defined (whenever they exist)+) by the strong limits

$$W_\pm = W_\pm(H, H_0) = \underset{t \to \pm\infty}{s\text{-}\lim} \; e^{tL_H} \cdot e^{-tL_{H_0}} \tag{6.2}$$

with respect to the $\mathcal{L}_1$ norm

$$\|f\| = \int_{\Gamma_{Phys}} |f(q,p)| \, \pi \, \frac{dq\,dp}{2\pi} \; . \tag{6.3}$$

This means that whenever f is a (say, smooth) function with a finite $\mathcal{L}_1$ norm, we have

$$\lim_{t \to \pm\infty} \left\| \left(W_\pm(H, H_0) - e^{tL_H} \cdot e^{-tL_{H_0}} \right) f \right\| = 0 \; . \tag{6.4}$$

Here q and p label the independent (physical) phase space coordinates. One should think, for example, of the variables $\underline{q}_a$ and $\underline{p}_a$ when the constraints are written in the canonical form (4.2) and the gauge condition $q^0_{ab} = 0$ (4.14) is adopted; in that case we have

$$L_{H^c} = L_h = \sum_a \left(\frac{\partial h}{\partial \underline{p}_a} \cdot \frac{\partial}{\partial \underline{q}_a} - \frac{\partial h}{\partial \underline{q}_a} \cdot \frac{\partial}{\partial \underline{p}_a} \right) \tag{6.5}$$

translation invariance implying $\sum_a \frac{\partial h}{\partial \underline{q}_a} = 0$.

The following easily verifiable intertwining property of the wave operators is characteristic for them:

$$e^{tL_H} W_\pm = W_\pm \, e^{tL_{H_0}} \; , \qquad L_H W_\pm = W_\pm L_{H_0} \; . \tag{6.6}$$

What is, however, important for our subsequent discussion is the possibility to reconstruct the total Hamiltonian H from the knowledge of H_0 and, say of w_+ in the absence of bound states.

+) We note that the existence of such operators is established for a wider class of Hamiltonians in the quantum theory, than in the classical framework (see, e.g., the discussion on existence of global solutions of the classical equations of motion in R1)). The existence problem for a classical S-matrix is studied by Hunziker H5) .

Let the Hamiltonian $H = H_o + \phi$ admit no bounded motion. Technically, this means that if $q(t)$, $p(t)$ is a solution of the equations of motion

$$\dot{q} = \frac{\partial H}{\partial p} \quad , \qquad \dot{p} = \frac{\partial H}{\partial q}$$

then

$$\lim_{t\to\infty} \phi(q(t), p(t)) = 0 \, . \tag{6.7}$$

(Physically, Eq. (6.7) is an expression of the hypothesis that particles get far apart for $t\to\infty$ and that the interaction is assumed to vanish for such asymptotic configurations.) This property will certainly be satisfied for (separable) repulsive potentials . Under this assumption we shall prove that

$$W_+ H_o = H \tag{6.8}$$

(W_+ could be replaced by W_- if the limit (6.7) were zero for $t \longrightarrow -\infty$).

First we note that since

$$e^{-tL_{H_o}} f = f - t\{f, H_o\} + \tfrac{1}{2} t^2 \{\{f, H_o\}, H_o\} + \cdots \tag{6.9}$$

we have $e^{-tL_{H_o}} H_o = H_o$ so that

$$e^{tL_H} \cdot e^{-tL_{H_o}} H_o = e^{tL_H}(H-\phi) = H - \phi(q(t), p(t));$$

taking the strong limit $t\to\infty$ (after applying both sides of the last equation to a smooth $\mathcal{L}_1$ -function $f(p,q)$) and using (6.7) we obtain (6.8).

It is straightforward to construct compatible constraints in terms of the (classical) wave operators in a Euclidean (more precisely $\mathbb{R}^1$ x E(3) where $\mathbb{R}^1$ is time translation) invariant canonical Hamiltonian theory for which the wave operator $W_+(h, h_o)$ exists, where in the equal time gauge

$$h = h(\underline{p}_1, \ldots, \underline{p}_N ; \underline{q}_{(12)}, \ldots, \underline{q}_{N-1\,N}) \quad , \qquad h_o = \sum_{a=1}^{N} \sqrt{m_a^2 + \underline{p}_a^2} \tag{6.10}$$

and h does not admit a bounded motion. To this end one sets

$$\varphi_a^c = W_+\left(\sqrt{m_a^2 + \underline{p}_a^2} - p_a^o\right) = W_+\left(\sqrt{m_a^2 + \underline{p}_a^2}\right) - p_a^o \approx 0 \, ; \quad a = 1, \ldots, N. \tag{6.11}$$

The compatibility condition follows from the identity

$$W_+(\{f,g\}) = \{W_+f, W_+g\}. \tag{6.12}$$

Eq. (6.12) is implied by the fact that e^{tL_H} generates a canonical transformation which commutes with the Poisson brackets:

$$e^{tL_H}(\{f,g\}) = \{e^{tL_H}f, e^{tL_H}g\}. \tag{6.13}$$

If W_+ is Poincaré invariant, we can also assert that the surface $\mathcal{M}$, given by the constraints (6.11), will be Poincaré invariant.

The problem is to construct a W_+ corresponding to 2-body forces. To this end we shall first write down an alternative representation for the 2-particle wave operators.

Let $h_{ab} = h_a + h_b$ be the canonical 2-particle Hamiltonian for particles a and b, where the constraints $\varphi_a^c = 0 = \varphi_b^c$ are assumed to satisfy conditions (i) (ii) (iii) of Section 4A. Let further

$$h_{ab} = h_{ab}^0 + V_{ab}, \quad \text{where} \quad h_{ab}^0 = \sqrt{m_a^2 + \underline{p}_a^2} + \sqrt{m_b^2 + \underline{p}_b^2}. \tag{6.14}$$

Then the operator $W_+^{(ab)} = W_+(h_{ab}, h_{ab}^0)$ can be written in the form

$$W_+^{(ab)} = \lim_{t\to\infty} e^{tL_{h_{ab}}} e^{-tL_{h_{ab}^0}} = T^* \exp \int_0^\infty L_{V_{ab}(t)}\, dt \tag{6.15}$$

where T stands for the antichronological product and

$$V_{ab}(t) = e^{tL_{h_{ab}^0}} V_{ab} \tag{6.16a}$$

so that

$$L_{V_{ab}(t)} = e^{tL_{h_{ab}^0}} L_{V_{ab}} e^{-tL_{h_{ab}^0}}. \tag{6.16b}$$

The "interaction picture formulas" (6.15) and (6.16) follow from the finite-time relation

$$e^{tL_{h_{ab}}} e^{-tL_{h_{ab}^0}} = T^* \exp \int_0^t L_{V_{ab}(t)}\, dt. \tag{6.17}$$

(In order to prove this last relation one verifies by differentiating with respect to t that both sides of Eq. (6.17) satisfy the same first order differential equation with the same initial condition.)

Proposition 6.1. Let, for a fixed Lorentz frame

$$V(t) = \sum_{1\le a<b\le N} V_{ab}(t); \tag{6.18}$$

then we define the N-particle wave operator

$$W_+ = T^* \exp \int_0^\infty L_{V(t)} \cdot dt \tag{6.19}$$

(assuming again that the right hand side makes sense as an operator on $\mathcal{L}_1$) . We claim that then the constraints (6.11) satisfy all conditions (i)-(iii) of Sec. 4A.

Sketch of the proof. Compatibility is a consequence of (6.12). Cluster follows from the additivity property

$$L_V = \sum_{a<b} L_{V_{ab}} \tag{6.20}$$

and from the assumed separability of V_{ab} (i.e. from $V_{ab} \longrightarrow 0$ for $r_{ab} = (q_{ab}^2 + (\hat{P}_{ab}\, q_{ab})^2)^{1/2} \longrightarrow \infty$). Lorentz invariance can be deduced from the Birmann-Kato invariance principle (cf. also the work of Sokolov S4), S6)).

6C. A series expansion of N-particle constraints in terms 2-body forces

We shall look for manifestly Poincaré invariant N-particle constraints

$$\varphi_a = \tfrac{1}{2}(m_a^2 + p_a^2) + \phi_a + \tilde{\phi}_a \quad , \quad a = 1, \ldots, N \tag{6.21}$$

where

$$\phi_a = \sum_{b=1}^{N} \phi_{ab} \qquad (\phi_{aa} = 0) \tag{6.22}$$

ϕ_{ab} is a 2-particle "potential" of type (6.1), and $\tilde{\phi}_a$ is a sum of n-particle interaction ($3 \leq n \leq N$) to be determined from the compatibility condition

$$\{\varphi_a, \varphi_b\} = p_a \frac{\partial}{\partial q_a} \tilde{\phi}_b - p_b \frac{\partial}{\partial q_b} \tilde{\phi}_a + \{\phi_a + \tilde{\phi}_a, \phi_b + \tilde{\phi}_b\} = 0 \tag{6.23}$$

Assuming that the 2-body terms ϕ_{ab} are proportional to a (small) parameter $\alpha_{ab} = \alpha$, we look for solutions $\tilde{\phi}_a$ of Eq. (6.23) that are second order in the α's .

Setting

$$\tilde{\phi}_a = \tilde{\phi}_a^{(2)} + \tilde{\phi}_a^{(3)} + \ldots \qquad \text{where} \qquad \tilde{\phi}_a^{(K)} = O(\alpha^K) \tag{6.24}$$

we deduce from (6.23) a recursive system of equations for $\tilde{\phi}_a^{(K)}$:

$$p_b \frac{\partial}{\partial q_b} \tilde{\phi}_a^{(2)} - p_a \frac{\partial}{\partial q_a} \tilde{\phi}_b^{(2)} + \{\phi_a, \phi_b\} = 0 \tag{6.25a}$$

$$p_b \frac{\partial}{\partial q_b} \overset{(3)}{\phi}_a - p_a \frac{\partial}{\partial q_a} \overset{(3)}{\phi}_b + \{\phi_a, \overset{(2)}{\phi}_b\} + \{\overset{(2)}{\phi}_a, \phi_b\} = 0 \qquad (6.25b)$$

$$p_b \frac{\partial}{\partial q_b} \overset{(4)}{\phi}_a - p_a \frac{\partial}{\partial q_a} \overset{(4)}{\phi}_b + \{\phi_a, \overset{(3)}{\phi}_b\} + \{\overset{(2)}{\phi}_a, \overset{(2)}{\phi}_b\} + \{\overset{(3)}{\phi}_a, \phi_b\} = 0 \qquad (6.25c)$$

etc.

A crucial device for finding an explicit solution of Eq. (6.25a) is the introduction of the 2-particle vector valued function B_{ab}, antisymmetric in a,b, which satisfies

$$p_a \frac{\partial}{\partial q_a} B_{ab} = p_b \frac{\partial}{\partial q_b} B_{ab} = \frac{\partial \phi_{ab}}{\partial q_a} \quad \left(= \frac{\partial \phi_{ab}}{\partial r_{ab}} \frac{\xi_{ab}}{r_{ab}}\right) \qquad (6.26)$$

where $B_{ab} = B_{ab}\ (\xi_{ab}, \pi_{ab}, w_{ab})$,

$$\xi_{ab} = q_{ab} + (\hat{p}_{ab} q_{ab})\hat{p}_{ab}; \quad \pi_{ab} = p_a + (\hat{p}_{ab} p_a)\hat{p}_{ab} = -\pi_{ba} \qquad (6.27)$$

$(r_{ab} = |\xi_{ab}|)$, w_{ab} and $\hat{p}_{ab}$ being defined in Eq. (6.1).

Given B_{ab} we verify that

$$\overset{(2)}{\phi}_a = \sum_{b \neq c} \left(\frac{\partial \phi_{ab}}{\partial p_a} B_{ac} + \frac{\partial \phi_{ab}}{\partial p_b} B_{bc} - \frac{1}{2} B_{ab} B_{ac} \right) \qquad (6.28)$$

is a solution of (6.25a) (as a consequence of (6.26)).

The functions B_{ab} can be constructed as follows:

$$B_{ab} = \phi_{ab} \frac{\pi_{ab}}{\pi_{ab}^2} + \frac{l_{ab}}{\pi_{ab}^2} F_{ab}\left(\frac{l_{ab}}{|\pi_{ab}|}, \frac{\pi_{ab} q_{ab}}{l_{ab}}\right)\left(\xi_{ab} - (\hat{\pi}_{ab} q_{ab})\hat{\pi}_{ab}\right) \qquad (6.29a)$$

$$\hat{\pi}_{ab} = \frac{1}{|\pi_{ab}|} \pi_{ab}$$

where

$$F_{ab}(u, z) = \int_0^z \frac{1}{u\sqrt{1+x^2}} \frac{\partial}{\partial r_{ab}} \phi_{ab}(u\sqrt{1+x^2}, w_{ab}) \cdot dx \qquad (6.29b)$$

and l_{ab} is the (relative) angular momentum of the system (ab):

$$l_{ab}^2 = \pi_{ab}^2 r_{ab}^2 - (\pi_{ab} \xi_{ab})^2. \qquad (6.29c)$$

We observe that the solution (6.28) for $\overset{(2)}{\phi}_a$ only involves 3-body forces (it is, therefore, consistent to postulate that n-body forces for $n \geqslant 4$ are of higher than second order with respect to the 2-particle coupling constant α). This solution is symmetric in the variables (q_b, p_b) $b \neq a$. As far as the B-function vanish for infinite particles' separation (provided that ϕ and $\frac{\partial \phi}{\partial r}$ tend to zero for $r \longrightarrow \infty$) it satisfies evidently the cluster decomposition requirement.

In the important special case of the first order in α contribution

to the (relativistic) Coulomb potential (5.31),

$$\phi^{(1)}_{ab} = \frac{\alpha_{ab}\, E_{ab}}{r_{ab}} \quad , \quad \alpha_{ab} = \frac{e_a e_b}{4\pi} \quad , \quad E_{ab} = \frac{w_{ab}^2 - m_a^2 - m_b^2}{2 w_{ab}} \tag{6.30}$$

we obtain the simple expression

$$B^{Coul}_{ab} = \phi^{(1)}_{ab}\ \frac{r_{ab}^2 \pi_{ab} - (\pi_{ab} n_{ab}) n_{ab}}{l_{ab}^2} \tag{6.31}$$

We end our discussion with a couple of remarks.

The solution of the system (6.25) is clearly not unique. At each step the functions $\tilde{\phi}^{(\kappa)}_a$ are determined up to a solution of the system of linear homogeneous equations

$$p_a \frac{\partial}{\partial q_a} \tilde{\phi}^{(\kappa)}_b - p_b \frac{\partial}{\partial q_b} \tilde{\phi}^{(\kappa)}_a = 0 \ . \tag{6.32}$$

The general solution of (6.32) is

$$\tilde{\phi}^{(\kappa)}_a = p_a \frac{\partial}{\partial q_a} \tilde{\phi}^{(\kappa)} \tag{6.33}$$

where $\tilde{\phi}^{(\kappa)}$ is an arbitrary smooth function of the phase space variables. The problem of finding optimal additional restrictions on $\tilde{\phi}_a$ (such as appropriate Cauchy data on certain surfaces), which would lead to a unique $\tilde{\phi}_a$ is, to our view, open. The "initial condition" adopted in S3) is not satisfied by our solution (6.29). We also have not discussed the relation between the wave operators approach of Sec. 6B and the special iterative solution of the present subsection. An application of Eq. (6.28) (or(6.31))to a realistic 3-particle problem with electromagnetic interaction (in the lines of B1)) may be the most practical way to overcome the remaining ambiguities.

7. Gauge dependence of canonical world lines and gauge invariance of asymptotic results

7A. Gauge dependence of canonical world lines for two interacting particles

We saw in Sec. 5 that there is a wide class of generalized 2-particle mass shells, which include good candidates for a realistic (manifestly covariant) 2-particle dynamics, say, in the Markov-Yukawa gauge (5.27). The question arises whether the "world lines" in the space of canonical coordinates q_1 and q_2 depend on the choice of the Lagrange multipliers λ_1 and λ_2 in the definition of the Hamiltonian

$$H(\lambda_1,\lambda_2) = \lambda_1\varphi_1 + \lambda_2\varphi_2 \approx 0.$$

The answer to this question is negative. For φ_a given by (5.1) (5.15) the canonical world lines are gauge invariant only in the case of zero interaction ϕ. (That is a consequence of the nontrivial dependence of the orthogonal distance r (5.16) on the total momentum P, which implies that

$$\{q_1, r\} = \frac{\partial r}{\partial P} = \frac{q\hat{P}}{rw}\, q_\perp = \{q_2, r\} \neq 0.)$$

More generally, the following negative result was established in M3).

Theorem 7.1. Let $\mathcal{M}$ be a generalized 2-particle mass-shell, satisfying conditions (i) and (ii) (but not necessarily (iii)) of Sec. 4A. The projection $\pi(\gamma_*)$ of each 2-dimensional fibre $\gamma_* \subset \mathcal{M}$ of the bundle $\mathcal{M} \to \Gamma_*$ into the (canonical) Minkowski space M of each particle ($\pi_a(q_1,p_1;q_2,p_2) = q_a$, $a = 1,2$) is a 1-dimensional submanifold of M_a iff $\mathcal{M}$ is (locally) physically equivalent to a free particle mass shell, so that the q-space trajectories of the particles are straight lines.

Sketch of the proof: In one direction the theorem is trivial. If the constraints are given by

$$\varphi_1^{free} = \frac{1}{2}\left[m_1^2 + (p_1 + qB(\tfrac{1}{2}q^2))^2\right] \approx 0,$$

$$\varphi_2^{free} = \frac{1}{2}\left[m_2^2 + (p_2 - qB(\tfrac{1}{2}q^2))^2\right] \approx 0 \quad , (q = q_1 - q_2) \tag{7.1}$$

i.e. if they are obtained from the free mass shell by a canonical transformation of type (4.12) or, more generally, if

$$\frac{\partial\varphi_1}{\partial p_2} = 0 = \frac{\partial\varphi_2}{\partial p_1}, \tag{7.2}$$

then q_1 does not depend on the "proper evolution parameter" σ_2 of the second particle and vice-versa:

$$\frac{\partial q_1}{\partial\sigma_2} = \{q_1, \varphi_2\} = 0 \quad , \qquad \frac{\partial q_2}{\partial\sigma_1} = \{q_2, \varphi_1\} = 0. \tag{7.3}$$

Hence, the projections $l_a = \pi_a\gamma_*$ of the fibre γ_* are 1-dimensional.

The converse statement (that the requirement

$$\dim l_a = 1 \tag{7.4}$$

implies (7.1) is both more interesting and more difficult to establish.

Its proof can be split in two steps. The first one is simple and works for the N-particle case.

Lemma 7.1. If the world lines are 1-dimensional, then

$$\frac{\partial \varphi_a^c}{\partial p_b} = \frac{\partial h_a}{\partial p_b} = 0 \qquad \text{for} \qquad a \neq b \tag{7.5}$$

(where φ_a^c are given by (4.2)).

Proof of Lemma 7.1. Choose $q_1^o, \ldots, q_N^o$ as evolution parameters on the world lines $\ell_1, \ldots, \ell_N$. Condition (7.4) implies that we can take these parameters as local coordinates in the fibre. Since ℓ_a is 1-dimensional, all tangent vectors to it are proportional; hence

$$\{q_a^\mu, \varphi_b^c\} = B_{ab} \frac{dq_a^\mu}{dq_a^o} \quad , \qquad a,b = 1, \ldots, N \quad \text{(no sum!)} \tag{7.6}$$

The coefficients B_{ab} may depend on the point $\gamma \in M$ but not on the Lorentz index μ. Since, according to (4.2),

$$\{q_a^o, \varphi_b^c\} = \delta_{ab} \tag{7.7}$$

it follows that $B_{ab} = \delta_{ab}$. Lemma 7.1. is proven.

The second step in the proof of Theorem 7.1. is much more involved and only works for $N = 2$.

Lemma 7.2. The canonical constraints (4.2), satisfying the Lorentz invariance condition (4.11) and compatibility along with (7.5), can be replaced by equivalent constraints of type (7.1).

The reader interested in the proof of this Lemma is referred to the original work of Molotkov and the author M3). We shall limit ourselves at this point to a couple of remarks.

The (omitted) proof of Lemma 7.2. could be extended to a N particle system provided that the N(2N-1) scalar products of the 2N-1 translation invariant vectors p_a and $q_{ab} = q_a - q_b$ are independent. In a D-dimensional space time this is only true for

$$D \geq 2N - 1 \tag{7.8}$$

Hence, for D=4, it only works for $N \leq 2$. (In a 2-dimensional space-time the proof of Lemma 7.2. does not work even in the 2-particle case. Moreover, an example given in Sec. IIC of ref. M3) demonstrates that there exists a class of non-trivial gauge invariant dynamics for D=2, involving a zero-mass particle.) For larger N (violating condition (7.8)) the following weaker result takes place.

Theorem 7.2. Let the generalized N-particle mass-shell satisfy condition (6.4) of Theorem 7.1 and let in addition the canonical Hamiltonian $h(\underline{p}_1,\ldots,\underline{p}_N;\underline{q}_{12},\ldots,\underline{q}_{N-1\,N})$ $(\approx \sum_a \underline{p}_a^2)$ be non-degenerate in the sense that

$$\det\left(\frac{\partial^2 h}{\partial p_a^i \partial p_b^j}\right) \neq 0 \qquad (i,j=1,2,3\ ;\ a,b=1,\ldots,N) \tag{7.9}$$

(the left-hand side standing for the determinant of the 3N x 3N matrix of second derivatives of the Hamiltonian).

Then the (canonical) Minkowski space trajectories of all particle are straight lines.

The rather technical proof of this theorem is a straightforward extension of a similar argument by Leutwyler L3).

Note that the Hamiltonian of a free zero mass particle violates the non-degeneracy condition (7.9).

Finally, we remark that the proof of Theorem 7.1. is local and uses smoothness in open neighbourhoods. Therefore, it does no apply to boundary points of the generalized mass shell. If we define the generalized 2-particle mass shell as

$$\mathcal{M}(R)=\left\{(q_1,p_1;q_2,p_2)\in\Gamma^2,\ p_a^0=\sqrt{m_a^2+\underline{p}_a^2},\ a=1,2,\ r^2=q_\perp^2\geq R^2(w)>0\right\}, \tag{7.10}$$

then we will have a gauge invariant description of elastic scattering of relativistic balls with piecewise straight world lines (see Sec. 1C of ref. M3), as well as T5)).

7B. Relation to the Currie-Jordan-Sudarshan (CJS) "no-interaction theorem".

Theorem 7.1. is the counterpart of the (by now classical) "no-interaction-theorem" of refs. C4), L3), H3). In order to elucidate the precise relation between the two results, we start with a concise formulation of the CJS statement of the problem and main theorem.

A CJS N-particle system is defined by a Poisson bracket realization of the Lie algebra of the Poincaré group in the phase space $\Gamma_C = R^{6N}$ spanned by the 3-dimensional particle coordinates $\underline{q}_a$ and momenta $\underline{p}_b$ (a,b = 1,...,N) and equipped with the canonical symplectic form

$$\omega_c = \sum_{a=1}^{N} d\underline{q}_a \wedge d\underline{p}_a \ . \tag{7.11}$$

The Euclidean generatores $\underline{P}$ and $\underline{J}$ $(= \frac{1}{2}\,\varepsilon_{ijk}J^{jk})$ are assumed to have the standard ("free") form (4.4) and (4.10), while the Lorentz boosts J^{oi} and the Hamiltonian $P^o = h$ are required to satisfy the

so called "world line condition" (in the terminology of C4) G2))

$$\{ J^{oi}, q_a^j \} = q_a^i \{ q_a^j, h \}, \quad a=1,\dots,N \; ; \; i,j=1,2,3. \tag{7.12}$$

A CJS system is called non-degenerate if the equation $\dot{\underline{q}}_a = \{ \underline{q}_a , h \} \equiv \underline{V}_a(\underline{q},\underline{p})$ can be solved with respect to the canonical momenta $\underline{p}_a$ (or, equivalently, if Eq. (7.9) takes place).

CJS Theorem C4), L3) . Every CJS 2-particle system, and every non-degenerate N-particle system for $N \geq 3$, is canonically equivalent to a free CJS system (with Hamiltonian $h = h_o \equiv \sum_a \sqrt{m_a^2 + \underline{p}_a^2}$).

Remark. The CJS theorem was originally established in four space-time dimensions. It fails in two dimensions unless one adds extra assumptions (C4), H3)) .

Given a generalized N-particle mass shell $\mathcal{M}$ there is a natural condition under which one can construct a CJS system. Assume that the intersection of $\mathcal{M}$ with the equal time gauge $q_a^{\,o} = t$, $a = 1,\dots,N$ coincides with R^{6N} (and is a global section of the fibre bundle $\mathcal{M} \longrightarrow \Gamma_*$). If we define the Lorentz boosts by

$$J^{oi} = -\sum_a h_a q_a^i \tag{7.13}$$

which is consistent with (4.10) and (4.2) for $t=0$, then the world line condition (7.12) is a consequence of our assumption (7.4) of gauge invariance of the world lines. Indeed, according to Lemma 7.1 ,

$\{ q_a , h_b \} = 0$ for $a \neq b$; hence,

$$\{ J^{oi}, q_a^j \} = \sum_b q_b^i \{ q_a^j, h_b \} = q_a^i \{ q_a^j, h_a \} = q_a^i \{ q_a^j, h \}$$

in accord with (7.12). Thus Theorem 7.1 can be obtained from the CJS theorem (using Lemma 7.1 and the above argument) provided that the intersection of $\mathcal{M}$ with the equal time gauge is R^{6N} .

7C. Non-canonical position variables and gauge invariance of asymptotic results.

Three ways out of the difficulty, signalled by the results of the preceeding subsection, have been discussed in the literature.

One, consists in introducing a privileged gauge, like the Markov-Yukawa condition (5.27). Then, of course, we shall have well defined Poincaré covariant world lines (the Hamiltonian being fixed up to an overall factor). The disadvantage of such an approach is that it seems to be at odds with the cluster decomposition property in the many particle case. (See, however, the discussion of this point in the Lagran-

gian approach, advocated by Professor Longhi in these Proceedings.)

The second one introduces non-canonical position variables

$$x_a = x_a(q_1, \dots, q_N ; p_1, \dots p_N) \qquad \text{such that}$$

$$\{x_a, \varphi_b^c\} = 0 \qquad \text{for} \qquad a \neq b \tag{7.14}$$

(wich thus would have gauge invariant world lines).

We assume in addition that x_a is a 4-vector which coincides with q_a asymptotically (for $\varphi_a^c \to \sqrt{m_a^2 + \underline{p}_a^2} - p_a^o$).

In the 2-particle case (in 4-dimensional space-time), when the generalized mass-shell is given by the constraints (5.1) (5.12), it is convenient to demand in addition that

$$x_a = q_a \qquad \text{for} \qquad Pq = 0 \quad , a = 1,2 \tag{7.15}$$

as proposed by Droz-Vincent D5) and Sazdjian S2). Then one can write x_a in the form

$$x_a = q_a + A_a \hat{P} + B_a \hat{r} + C_a (p - (p\hat{r})\hat{r}) \quad ; \quad a = 1,2 \tag{7.16a}$$

where p is the relative momentum (satisfying $p P \approx 0$) , and

$$\hat{P} = \frac{1}{w} P \quad , \quad \hat{r} = \frac{1}{r} q_\perp \qquad (-\hat{P}^2 = 1 = \hat{r}^2) \tag{7.16b}$$

while the coefficients A_a , B_a , C_a can be written as power series in the variable

$$\chi = \frac{\hat{P} q}{w} \tag{7.17}$$

such that $A_a = O(\chi)$, $B_a = O(\chi^2)$, and $C_a = O(\chi^3)$ (for $\chi \to 0$) see N1).

Such a distinction between physical positions and canonical coordinates is justified -as explained already on the example of a free spinning particle in Sec. 3 . It, however, leaves little of the original simplicity of the constraint Hamiltonian approach. Therefore, it is important to realize that one can extract gauge invariant (and physically interesting) asymptotic results directly from the canonical scheme.

We have already introduced in Sec. 6 the classical wave operators $W_\pm (H, H_o)$. The reparametrization (or gauge) invariance is one of their basic properties. In the 2-particle case, for $\mathcal{M}$ given by (5.1) (5.15), it is a consequence of the elementary identity

$$e^{tL_{H+\lambda\varphi}}\, e^{-tL_{H_0+\lambda\varphi}} = e^{tL_H}\, e^{-tL_{H_0}}$$

valid for $\{H,\varphi\} = 0 = \{H_0,\varphi\}$. In general, this is the physical content of the Birmann-Kato invariance principle which says that for a wide class of smooth monotonously increasing functions $F(\xi)$ on the reals (such that $F'(\xi) > 0$ everywhere)

$$W_\pm(H,H_0) = W_\pm(F(H),F(H_0)) . \tag{7.18}$$

(This principle, originally established in the quantum mechanical framework, was justified in the classical context by Sokolov S5).) The gauge invariance of the classical scattering operator is then a consequence of the relation

$$S = W_+^* W_- . \tag{7.19}$$

The gauge invariant asymptotic results also include the quantum mechanical bound state energy levels (which appear as poles of the scattering amplitude).

8. Quasipotential approach to the two-body problem in quantum electrodynamics.

8A. Quantization of the relative 2-particle motion

In order to make contact between the constraint Hamiltonian formalism and the quasipotential approach to the relativistic 2-body problem (developped in L6, 7) T3) F2) R3,4) it is sufficient to consider the quantization of the relative 2-particle motion, regarding the total momentum P as an external parameter (analogous to the energy E in the stationary non-relativistic Schrödinger equation). We shall restrict our attention in the present lectures to this simple part of the problem of quantization. One way to deal with the much more complicated general problem is described in the work of Droz-Vincent, and of Horwitz and Rohrlich, presented at this Workshop.

Consider the (locally convex) topological vector space $\mathcal{S}_P$ of infinitely smooth functions $\Psi(q) = \Psi(q,P)$ which decrease fast in the variable $q_\perp = q + (q\hat{P})\hat{P}$ and satisfy the differential equation

$$P\cdot\frac{\partial}{\partial q}\Psi(q,P) = 0 \qquad (P^0 > |\underline{P}|) . \tag{8.1}$$

Here $q = q_1 - q_2$ is the relative canonical coordinate; as we know, the general solution of Eq. (8.1) is a function of $q_\perp$ and P .

Let n be an arbitrary time-like vector normalized by

$$|n\hat{P}| = 1 \qquad \left(\hat{P} = \frac{1}{w} P \quad , \quad \hat{P}^2 = -1\right) \tag{8.2}$$

then the scalar product

$$(\Phi, \Psi) = (\Phi, \Psi)_{\hat{P}} = \int \overline{\Phi}(q) \Psi(q) \cdot \delta(nq) d^4q \tag{8.3}$$

does not depend on n. Indeed, every solution of Eq. (8.1) can be written in the form

$$\Psi(q,P) = (2\pi)^{-3} \int \underset{\sim}{\Psi}(p,P) \cdot \delta(p\hat{P})\, e^{ipx} \cdot d^4p \; ; \tag{8.4}$$

inserting (8.4) into (8.3) we obtain the manifestly n-independent expression

$$(\Phi, \Psi) = (2\pi)^{-3} \int \overline{\underset{\sim}{\Phi}(p,P)} \cdot \underset{\sim}{\Psi}(pP) \cdot \delta(p\hat{P})\, d^4p \, . \tag{8.5}$$

Boundstate wave functions (corresponding to discrete "eigenvalues" of $w = \sqrt{-P^2}$) are vectors in the Hilbert space completion $\mathcal{H}_P$ of $\mathcal{S}_P$.

We say that a dynamical variable is a relative observable if it commutes with the operator $P\frac{\partial}{\partial q}$ in the left hand side of equation (8.1). The variable $q_\perp$, the orthogonal (to P) coordinate difference, is such an observable and so is the relative momentum operator

$$p = -i \frac{\partial}{\partial q} \left[= -i\left(\frac{\partial}{\partial q} + \hat{P} \cdot \left(\hat{P} \cdot \frac{\partial}{\partial q}\right)\right) \text{ on solutions of (8.1)} \right]. \tag{8.6}$$

They satisfy the commutation relations

$$[p_\mu, p_\nu] = 0 = [q_\perp^\mu, q_\perp^\nu] \quad , \quad [q_\perp^\mu, p_\nu] = i\pi^\mu_\nu \qquad \left(\pi^\mu_\nu = \delta^\mu_\nu + \hat{P}^\mu \hat{P}_\nu\right). \tag{8.7}$$

The space $\mathcal{H}_P$ is, so to speak, the leaving room for all systems whose generalized mass shell lies on the hypersurface $pP = 0$. For a given interaction (an energy and angular momentum dependent "quasipotential") $\phi(r, w, l)$ $(r = |q_\perp|)$ we shall write the stationary Schrödinger type equation

$$H\Psi \equiv \left[\frac{1}{2}(p^2 - b^2(w)) + \phi(r,w,l)\right] \Psi(q,P) = 0 \, . \tag{8.8}$$

In the centre of mass frame, setting

$$P = (w, 0) \quad , \quad q \to q_\perp = (0, \underline{r}) \quad (r = |\underline{r}|), \quad p = (0, \underline{p}) = \left(0, -i\frac{\partial}{\partial \underline{r}}\right) , \tag{8.9}$$

we arrive at the 3-dimensional equation

$$\left[-\frac{1}{2}\left(\Delta + b^2(w)\right) + \phi(r,w,l)\right]\Psi_w(\underline{r}) = 0 \qquad (8.10a)$$

which is equivalent to a local differential equation for the partial waves $\Psi_l(r,w)$:

$$\left\{-\frac{1}{2}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + b^2(w) - \frac{l(l+1)}{r^2}\right] + \phi(r,w,l)\right\}\Psi_l(r,w) = 0. \qquad (8.10b)$$

8B. Reconstruction of the quasipotential for a given Feynman expansion of the scattering amplitude

The question arises: can we construct an interaction function such that the Schrödinger equation (8.10) would reproduce, e.g., the correct energy bound-state level corresponding to a given (local) quantum field theory? The answer is yes and the route to the construction passes through a relativistic Lippmann-Schwinger equation for the 2-particle scattering amplitude.

Consider the elastic scattering of two particles of masses m_1 and m_2, with inital momenta $p_1^{in} = p_1$, $p_2^{in} = p_2$ and final momenta p_1^f, p_2^f. The invariant scattering amplitude T is defined in terms of the 2-particle S-matrix element as follows:

$$\langle p_1^f, p_2^f | S | p_1, p_2 \rangle = \langle p_1^f | p_1 \rangle \langle p_2^f | p_2 \rangle + \\ + (2\pi)^4 i\, \delta(p_1^f + p_2^f - p_1 - p_2)\, T(p_1^f, p_2^f; p_1, p_2) \qquad (8.11)$$

where we use the following invariant normalization for the 1-particle states:

$$\langle p_a^f | p_a \rangle = (2\pi)^3 \cdot 2p_a^0 \cdot \delta(\underline{p}_a - \underline{p}_a^f) \;; \qquad a = 1,2 \qquad (8.12)$$

The symmetric (partial) off mass shell extrapolation of T to momenta satisfying

$$p_1^2 - p_2^2 = (p_1^f)^2 - (p_2^f)^2 = m_2^2 - m_1^2 \qquad (8.13)$$

can be regarded as a function of the total momentum

$$P = p_1 + p_2 = p_1^f + p_2^f \qquad (8.14)$$

and the two (orthogonal to P) relative momenta

$$p = \mu_2 p_1 - \mu_1 p_2 \;\; ; \;\; p^f = \mu_2 p_1^f - \mu_1 p_2^f \;\; ; \;\; \mu_{1,2} = \frac{1}{2} \pm \frac{m_1^2 - m_2^2}{8w^2} \quad (Pp = 0 = Pp^f). \tag{8.15}$$

We postulate for T the following (relativistic) Lippmann-Schwinger type equation:

$$T_P(p^f, p) + K_P(p^f, p) + \\ + (2\pi)^{-3} \int K_P(p^f, k) \cdot G_P(k) \cdot T_P(k, p) \cdot \delta(2Pk) \cdot d^4k = 0 \tag{8.16}$$

where G_p is the Green function

$$G_P(k) = \frac{1}{k^2 - b^2(w) - i0} \,. \tag{8.17}$$

This choice of $G_p(k)$ is imposed by the following two requirements (a) We demand that for a hermitian K ,

$$K^* = K \quad \text{where} \quad K_P^*(p, k) = \overline{K_P(k, p)} \,. \tag{8.18}$$

Eq. (8.16) should imply the elastic unitary condition

$$T_P(p^f, p) - T_P^*(p^f, p) = \frac{i}{(2\pi)^2} \int T_P^*(p^f, k) \cdot T_P(k, p) \cdot \delta(2Pk) \cdot \delta(k^2 - b^2(w)) \cdot d^4k \,, \tag{8.19}$$

(b) We require that $G_P^{-1}(p)$ is a second order (Poincaré invariant) Local differential operator (for $p = -i \frac{\partial}{\partial q}$) .

Under this assumption the corresponding homogeneous equation will be of type (8.8) (or (8.10)).

The implication (8.18) $\Rightarrow$ (8.19) determines the discontinuity of the Green function. Indeed, writing equation (8.16) and its solution in the symbolic form

$$T + K + K G T = 0 \tag{8.16'}$$

$$T = -\frac{1}{1+KG} K = -K \frac{1}{1+GK} \tag{8.20}$$

we find (using $K = K^*$)

$$T - T^* = \left(K \frac{1}{1+GK}\right)^* - \frac{1}{1+KG} K = \\ = \frac{1}{1+KG^*} K (G - G^*) \frac{1}{1+KG} K = T^* (G - G^*) T \,. \tag{8.21}$$

For G given by (8.17) the discontinuity is

$$G_{\underline{p}}(k) - G_{\underline{p}}(k)^* = 2\pi i\, \delta(k^2 - b^2). \tag{8.22}$$

Inserting (8.22) in (8.21) and deciphering the short hand notation in the right-hand side we recover (8.19).

In the centre of mass frame (8.9), setting $T_{(w,\underline{Q})}(o, \underline{p}^f ; o, \underline{p}) = T_w(\underline{p}^f, \underline{p})$ etc., we arrive at the following 3-dimensional form of the relativistic Lippmann-Schwinger equation:

$$T_w(\underline{p}^f, \underline{p}) + K_w(\underline{p}^f, \underline{p}) + \frac{1}{2w}\int K_w(\underline{p}^f, \underline{k}) \cdot \frac{1}{\underline{k}^2 - b^2(w) - i0} \cdot T_w(\underline{k}, \underline{p}) \cdot \frac{d^3k}{(2\pi)^3} = 0 . \tag{8.23}$$

In order to exhibit the precise relation between Eq. (8.23) and the homogeneous Schrödinger-like equation of type (8.8), we introduce the wave function $\underset{\sim}{\psi}_{\underline{p}}(\underline{k})$ corresponding to the scattering problem:

$$\underset{\sim}{\psi}_{\underline{p}}(\underline{k}) = (2\pi)^3 \delta(\underline{p} - \underline{k}) + \frac{1}{2w_{\underline{p}}(\underline{k}^2 - \underline{p}^2 - i0)} \cdot T_{w_{\underline{p}}}(\underline{k}, \underline{p}) \tag{8.24a}$$

where

$$w_{\underline{p}} = \sqrt{m_1^2 + \underline{p}^2} + \sqrt{m_2^2 + \underline{p}^2} . \tag{8.24b}$$

Inserting in (8.23), we find

$$(\underline{p}_f^2 - \underline{p}^2)\, \underset{\sim}{\psi}_{\underline{p}}(\underline{p}^f) + \frac{1}{2w_{\underline{p}}}\int K_{w_{\underline{p}}}(\underline{p}^f, \underline{k}) \cdot \underset{\sim}{\psi}_{\underline{p}}(\underline{k}) \cdot \frac{d^3k}{(2\pi)^3} = 0 \tag{8.25}$$

In the case of a local quasipotential, for

$$K_w(\underline{p}, \underline{k}) = K_w(\underline{p} - \underline{k}) \tag{8.26}$$

(which has its part in the applications), Eq. (8.25) becomes a special case of (8.8) (or (8.10)) for

$$2\phi(r, w) = \frac{1}{2w}\int K_w(\underline{p})\, e^{i\underline{p}\cdot\underline{r}} \frac{d^3p}{(2\pi)^3} = \int K_{\underline{p}}(p) \cdot \delta(2\mathcal{P}p)\, e^{ipq_\perp} \cdot \frac{d^4p}{(2\pi)^3} . \tag{8.27}$$

The question of determining the interaction term ϕ in the (quantized) generalized mass-shell equation is thus reduced to finding the

kernel K in the Lippmann-Schwinger equation. On the other hand, for given perturbation expansion of the scattering amplitude

$$T = T_1 + T_2 + \dots \tag{8.28a}$$

(where T_n is the term of order n in the expansion parameter), having, for example, the Feynman diagram expansion of T for a given quantum field theoretic Lagrangian, we can use Eq. (8.16) to obtain a corresponding expansion

$$K = K_1 + K_2 + \dots \tag{8.28b}$$

for the kernel K . Inserting (8.28) in (8.16), we find

$$K_1 = -T_1 \quad , \quad K_2 = -T_2 + T_1 G T_1 \quad , \quad \text{etc.} \tag{8.28c}$$

With such a definition of K it would of course be a vicious circle to try to solve (8.16) with respect to T in a perturbative way. However, if we are able to solve (8.16) or (8.25) (resp. (8.10)) exactly (or more generally, in a non-perturbative way) even for $K = K_1$ we will obtain an information about T (or Ψ) which is not contained in any finite number of terms of the perturbation expansion (8.28a) . We shall see in the next subsection how one can obtain a relativistic Balmer formula in this way.

8C. Effective 4-potential for hydrogen-like systems. Fine structure of the energy spectrum

In order to obtain a realistic application of the quasipotential equation we would like to determine in the lowest order in perturbation theory the effective potential V^{μ} that enters the stationary (relativistic) Schrödinger equation

$$H_{em}\Psi \equiv \frac{1}{2}\left[m_w^2 + (\underline{p}-\underline{V})^2 - (E-V^0)^2\right]\Psi = 0 \tag{8.29}$$

which is the quantized version of the Hamiltonian constraint (5.30) (m_w and E being given by (5.22) and (5.25), respectively). Using (8.27) and (8.28c) we would like to express the momentum space counterpart of the interaction function

$$2\phi = 2EV^0 - \{\underline{p},\underline{V}\} + \underline{V}^2 - V_0^2 \tag{8.30}$$

in terms of the Born diagram in the theory of two oppositely charged spinless particles. Keeping the first order in α terms in (8.30) we find

$$2E\,V^0(\underline{p},\underline{k}) - (\underline{p}+\underline{k})\,\underline{V}(\underline{p},\underline{k}) = -\frac{1}{2w}\,T_1(\underline{p},\underline{k}) =$$

$$= -\frac{e^2}{2w}\cdot\frac{4E_1E_2 + (\underline{p}+\underline{k})^2}{(\underline{p}-\underline{k})^2} \qquad (8.31)$$

where

$$4E_1E_2 = 4\mu_1\mu_2 w^2 = w^2 - \frac{(m_1^2-m_2^2)^2}{w^2} = 4\left(Ew - b^2(w)\right) \qquad (8.32)$$

($E_a = \mu_a w$, $a = 1,2$).
(We note that the on-shell expression for T_1 in the right-hand side of (8.31) is gauge independent.)

It is clear that the 4-potential V^μ is not determined uniquely from (8.31), since adding to it any 4-vector orthogonal to $(2E, \underline{p} + \underline{k})$ would not change the left-hand side of the equation. This gauge freedom can be used in order to identify V^0 with the Coulomb potential:

$$V^0(\underline{p},\underline{k}) = -\frac{e^2}{(\underline{p}-\underline{k})^2} \qquad (8.33)$$

Using the last equation (8.32) and noting that on the mass shell

$$(\underline{p}+\underline{k})^2 - 4b^2 = (\underline{p}-\underline{k})^2$$

we find

$$(\underline{p}+\underline{k})\cdot\underline{V}(\underline{p},\underline{k}) = -\frac{e^2}{2w}\,.$$

For an appropriate choice of the 3-dimensional gauge we can write the following expression for the space part of the vector potential:

$$\underline{V}(\underline{p},\underline{k}) = -\frac{e^2}{2w}\cdot\frac{\underline{p}+\underline{k}}{(\underline{p}+\underline{k})^2}\,. \qquad (8.34)$$

Finally, we extrapolate the validity of (8.33) off mass shell (dropping the condition $\underline{p}^2 = \underline{k}^2 = b^2(w)$). In the $\underline{r}$-space picture V^μ is given by the pseudo-local expression

$$V^0 = -\frac{\alpha}{r}\,, \qquad \underline{V} = -i\,\frac{\alpha\,\underline{r}}{2w\,r^3}\cdot I_s \qquad \left(\alpha = \frac{e^2}{4\pi}\right) \qquad (8.35)$$

where I_s is the space reflection operator: $I_s\,\underline{\psi}(\underline{r}) = \underline{\psi}(-\underline{r})$.
The effective potential (8.30) turns out to be a local multiplication operator, since

$$\underline{p}\cdot\underline{V} + \underline{V}\cdot\underline{p} = -\frac{2\pi\alpha}{w}\,\delta(\underline{r}) \qquad \left(\underline{p} = -i\underline{\nabla}\right) \qquad (8.36a)$$

$$\underline{V}^2 = \frac{\alpha^2}{4w^2r^4}\,. \qquad (8.36b)$$

This completes the evaluation of the operator H_{em} in (8.29) for spinless charged particles and can be used for calculating the fine structure corrections in that case (see T3) R3)). By the same type of reasoning one can also derive the following expression for the case when particle 1 has spin $\frac{1}{2}$, while particle 2 is still spinless (see R4)):

$$H_{em} = -\frac{1}{8}\left(\Delta + b^2(\omega)\right) - E\frac{\alpha}{r} - \frac{1}{2}\frac{\alpha^2}{r^2} + \pi\frac{\alpha}{\omega}\delta(\underline{r}) + \frac{\alpha^2}{8\omega^2 r^4} + \\ + \pi\frac{\alpha}{\omega}\frac{E_2}{E_1+m_1}\delta(\underline{r}) + \frac{\alpha}{2\omega r^3}\left(1+\frac{E_2}{E_1+m_1}\right)\underline{L}\,\underline{\sigma} \ . \qquad (8.37)$$

Here E_1 and E_2 are the centre of mass energies of particles 1 and 2 (given in (8.32)), $\underline{L} = \underline{r}\wedge\underline{p} = -i\underline{r}\wedge\underline{\nabla}$ is the orbital angular momentum, σ_j are the Pauli matrices. (The terms in the second line of the right-hand side of (8.37) disappear for spinless particles.) A standard evaluation of the energy eigenvalues leads to the following result:

$$w_{n\ell j} = M - \frac{m\alpha^2}{2n^2} + \frac{m\alpha^4}{8n^4}\left(3-\frac{m}{M}\right) - \frac{m\alpha^4}{Mn^3}\left(\frac{m_1-m}{2\ell+1} + \frac{m_2+m}{2j+1}\right) + \\ + O(\alpha^4) \qquad \left(M=m_1+m_2\ ,\ Mm = m_1m_2\ ,\ \ell = j\pm\tfrac{1}{2}\right). \qquad (8.38)$$

The right-hand side of (8.38) gives the correct expression for the energy eigenvalues up to order α^4 . This has been verified by evaluating higher order Feynman diagrams contributions to the quasipotential (see R4)). The possibility of obtaining the correct fine structure of the energy spectrum using only the single photon exchange diagram justifies a posteriori our choice of gauge which led us to a local expression for the effective potential (8.30).

9. Concluding remarks

In the preceding pages we have argued that there exists at present a self-consistent relativistic particle mechanics, both classical and quantum, which clarifies old puzzles and provides a background for practical 2-particle calculation. This does not mean, of course, that the subject is essentially closed. We would like to mention here some problems which require (and merit) additional study.

There have been two ways of deriving an expression for the electromagnetic forces between two (point) charged particles. One, pursued by Bel and colaborators B3-5), starts from classical electrodynamics and the corresponding retarded interaction. The second extracts the same type of information from quantum electrodynamics in the framework of the quasipotential approach (reviewed in Sec. 8 of these notes).

A systematic comparison of the results of these two approaches should be instructive.

The study of three and more particle dynamics still leaves much to desire. There is no coherent treatment of the problem of scattering on (and destroying) a bound state in the constraint Hamiltonian approach. The quantization of 3-particle interactions of type (6.28) (6.29) poses a non-trivial ordering problem. (That is a part of the general problem of quantizing constraint dynamical systems which is still in its infancy.)

Application of the quasipotential approach with a Richardson type of interaction R2) undertaken by Crater C3) (see also A1)) seems quite promising and should be pursued in a more systematic way.

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